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Dielectric behavior of anisotropic inhomogeneities: interior and exterior point Eshelby tensors

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Abstract
In this work we analyze the problem of finding the electric behavior of an anisotropic ellipsoid (arbitrarily shaped) placed in a dielectric anisotropic environment. We suppose that the whole system is exposed to a uniform electric field remotely applied. In order to find the resulting electric quantities inside the particle and outside it we adopt a technique largely utilized for solving similar problems in elasticity theory. The inhomogeneity problems in elastostatics are solved within the framework of the Eshelby theory, which adopts, as crucial points, the concepts of *eigenstrains* and *inclusions*. The generalization and assessment of such an approach for the dielectric inhomogeneity problems is here addressed by means of the introduction of the concepts of *eigenfields* and *inclusions* in electrostatics. The advantages of this methodology are mainly two: firstly, we can consider completely arbitrary dielectric anisotropic behavior both for the particle and the host matrix. Secondly, we easily find explicit expressions for the electric quantities both inside and outside the inhomogeneity. The problem under consideration was solved in earlier literature by analyzing the singularity of the dyadic Green function, expressed as a two-dimensional integral. Here we propose a reformulation described by a one-dimensional integral obtained from explicitly electrostatic analysis, which can have both pedagogical and computational importance. We also introduce a method to generalize these results to the case of an arbitrary nonlinear anisotropic ellipsoid embedded in a linear anisotropic matrix. Finally, we show some applications to the dielectric characterization of anisotropic composite materials.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

One of the most important preliminary issues in homogenization theory for composite materials is knowledge of the behavior of a single particle embedded in a given matrix. Once this problem is solved, the homogenization procedure continues by means of some ad hoc averaging of the electrical quantities over the whole region occupied by the heterogeneous material [1]. Typically, when an averaging process is chosen it generates a particular effective mean field theory [1, 2]. This approach has successfully been applied in several theories: one of the first attempts to model a mixture is given by the Maxwell theory for describing a strongly diluted suspension of spheres [1–3]. A better model is provided by the differential method, which derives from the mixture characterization approach used by Bruggeman [4]. In this case, the relations should maintain the validity also for less diluted suspensions. Moreover, the first papers dealing with mixtures of ellipsoids were written by Fricke [5], dealing with electrical characterization of inhomogeneous biological tissues containing spheroidal particles. Some other generalizations for a population of ellipsoids, obtained with the differential method, can be found in the current literature [6]. Usually, in the technical literature Maxwell's relation for spheres and Fricke's expressions for ellipsoids are so-called Maxwell–Garnett effective medium theory results [6, 7]. Recently important results have been obtained for complex [8, 9] and nonlinear dispersions of particles [10–14].

As said above, the starting point of all these theories is knowledge of the electrical behavior of a single particle embedded in the given matrix: the solution of this problem, formulated for isotropic ellipsoidal particles embedded in an isotropic environment, is well known and largely spread in applications. Relative mathematical expressions can be found in several textbooks on electromagnetism [15, 16]. The aim of the present work is to introduce a methodology to cope with the completely anisotropic problem and to illustrate the relative outcomes. In particular, we have found explicit expressions for the electric field both inside and outside the ellipsoidal particle. We have followed an approach that is widely utilized in similar problems within the elasticity theory. The problem of an inhomogeneity in heterogeneous elastic materials has been completely solved by Eshelby in the case of isotropic environment by means of a very elegant mathematical procedure [17, 18]. In this work we follow that line of thought but we apply it to anisotropic dielectric composite systems. The complete development of the Eshelby theory for the elasticity theory can be found in Mura's textbook [19], where all the details are deeply analyzed. As described above for the dielectric case, also for elastic materials knowledge of the behavior of a single inhomogeneity has opened the way to the characterization of composite elastic materials from the mechanical point of view (micro- and nano-mechanics) [20, 21]. Anyway, to make the paper suitable for a broad readership we do not use any results of the elastic Eshelby theory and we present all the detailed proofs within the electrostatic theory: so the development remains self-contained. Nevertheless, it is interesting to remark the strong analogy between electrostatics and elastostatics: in table 1 we draw a comparison among all the corresponding elastic and electric quantities for the reader interested in such a similarity. The electric equations are written in the absence of free charges and the elastostatic relations are written in the absence of external forces: the analogy holds on when all the sources are absent or remotely applied. For brevity, we do not discuss here the meaning of all the elastic quantities, which are explained in standard elasticity textbooks [22]. The analogy between elasticity and conductivity problems has been formalized in earlier works in order to homogenize composite materials by obtaining the effective elastic moduli and the effective conductivity [23]. In particular, it has been exactly proved that the effective conductivity corresponds to the effective bulk modulus, if the Poisson ratios of the phases are set to zero [23].

Table 1. Comparison among the most important quantities and basic equations of the electrostatics and the elasticity theory.

Electric quantities			Elastic quantities		
Electric field	E_i	$E_i = -\frac{\partial V}{\partial x_i}$	Elastic strain	$\epsilon_{s,ij}$	$\epsilon_{s,ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$
Electric potential	V		Elastic displacement	u_i	
Electric displacement	D_i		Stress tensor	T_{ij}	
Maxwell equation	$\frac{\partial D_i}{\partial x_i} = 0$		Balance of forces	$\frac{\partial T_{ij}}{\partial x_i} = 0$	
Permittivity tensor	ϵ_{ij}		Stiffness tensor	C_{ijkh}	
Constitutive equation	$D_i = \epsilon_{ij} E_j$		Constitutive equation	$T_{ij} = C_{ijkh}\epsilon_{s,kh}$	

The application of the Eshelby approach to several inhomogeneity problems is not new and many results can be found in the literature. In fact, a similar attempt, based on the equivalent inclusion method, has been developed in the framework of steady-state thermal conduction [24]. This approach has been successfully applied to the determination of the effective thermal conductivity of a misoriented short fiber composite and to the thermal conductivity of coated filler composites [25, 26]. In these works, the thermal Eshelby tensor has been obtained in a completely isotropic environment: we observe that our anisotropic result given in equation (32) (see below) is in perfect agreement with such achievements when isotropy is assumed. Moreover, electromagnetic characterization of different composite materials has been performed by means of the so-called depolarization tensor which can be simply related to our electric Eshelby tensor [11]. It has been introduced in the form of a dyadic Green function in a bianisotropic medium [27]. In particular, such a depolarization tensor has been applied to homogenize very general bianisotropic-in-bianisotropic particulate composites [11]. In addition, this method has been utilized to obtain the effective properties of nonlinear (or weakly nonlinear) heterogeneous media [12–14].

Our extensions to earlier methodologies, described in the present paper, are the following. In previous works the depolarization tensor for anisotropic and bianisotropic media is typically written in terms of a double integral (over the angles of a spherical coordinate system) [11]: by means of a completely different method, in this work we are able to present an equivalent internal electric Eshelby tensor (for anisotropic but not bianisotropic media) in the form of a simple integral (rapidly convergent for any kind of anisotropy, see equation (32)). This simplification is an improvement that can be useful to develop efficient numerical procedures for homogenizing composite materials. In other words, we provide a reformulation of the singularity of the dyadic Green function [27], known as a two-dimensional integral, as a one-dimensional integral with pedagogical and computational applications, which is obtained from explicitly electrostatic analysis. Moreover, we also present the external electric Eshelby tensor (see equation (33)) which completely governs the behaviour of the electric quantities around the embedded particle (in the entire space): to the authors' knowledge, an exact solution for the external fields in a complete anisotropic environment is not present in the previous literature. In fact, a common feature of all effective medium theories is that the actual microscopic distribution of the physical quantities around the particles are simply coarse grained. Nevertheless, the external Eshelby tensor facilitates the determination of the spatial distribution of the electric field nearby the inhomogeneity and evaluation of its local fluctuations, which are very useful to analyze material failure or breakdown phenomena [28]. These effects occur at localities where the intensity of the electric field is maximum or at spots where the energy concentration is very large [29]. Finally, we describe the generalization of the procedure to the case of a nonlinear anisotropic dielectric ellipsoid embedded in an anisotropic matrix. The result stated in equation (44) represents an implicit equation giving the exact electric field induced into such an inhomogeneity. Typically, electrically nonlinear problems have been solved approximately only for Kerr-like nonlinearities [13]. Therefore, our approach can be used to extend the homogenization procedures to arbitrarily anisotropic and nonlinear particles dispersions.

The paper is structured as follows: in section 2 we derive the Green function for the electrostatics of anisotropic media. So, in section 3, we introduce the concepts of eigenfield and inclusion. Moreover, we separately analyze the electric field due to an ellipsoidal inclusion (section 4) inside and outside the ellipsoid. The corresponding Eshelby tensors are described in section 5. At this point we explain the equivalence principle, which allows us to solve the inhomogeneity problem by using the concept of inclusion (both for linear particles in section 6 and nonlinear particles in section 7). Some computational aspects of the external fields are discussed in section 8. Finally, we show some applications to the dielectric characterization of anisotropic composite materials in section 9. It must be underlined that from a merely mathematical standpoint, the problem of calculating the effective permittivity tensor is identical to a number of others, for instance to that regarding permeability (in a magnetostatic situation), conductivity (in the stationary case), thermal conductivity (in a steady-state thermal regime), diffusivity (in stationary diffusion processes) and so on. Throughout all the paper we use the following notations: the vectors are indicated by $\vec{v}$ and tensors by $\overset{\leftrightarrow}{T}$. Moreover, we adopt the Einstein convention for implicit sums over repeated indices unless the explicit summation symbol is used.

2. Green function for anisotropic media

An important preliminary issue for the following purposes is the determination of the electric Green function for an anisotropic environment. Therefore, we have to solve the basilar equation of the electrostatics, as described in table 1, with an impulsive source corresponding to a charged point Q. It means that we have to solve the following differential problem,

$$\vec{\nabla} \cdot [\overset{\leftrightarrow}{\epsilon} \vec{\nabla} V(\vec{r})] = -Q\delta(\vec{r}) \tag{1}$$

or

$$\epsilon_{kl} \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_l} V(\vec{r}) = -Q\delta(\vec{r}), \tag{2}$$

with an arbitrary permittivity tensor $\overset{\leftrightarrow}{\epsilon}$ having elements ϵ_{kl} ($\delta(\vec{r})$ is the Dirac delta function). This problem is solved in appendix A by obtaining

$$V(\vec{r}) = \frac{Q}{4\pi\sqrt{\det \overset{\leftrightarrow}{\epsilon} [\vec{r}^T \overset{\leftrightarrow}{\epsilon}^{-1} \vec{r}]}}. \tag{3}$$

We now consider an arbitrary spatial charge distribution $\rho_{\mathrm{TOT}}(\vec{r})$. The corresponding electrostatic total potential $V_{\mathrm{TOT}}(\vec{r})$ in the anisotropic environment can be obtained with a convolution integral between the above Green function and charge distribution

$$V_{\mathrm{TOT}}(\vec{r}) = \frac{1}{4\pi\sqrt{\det \overset{\leftrightarrow}{\epsilon}}} \int_{\Re^3} \frac{\rho_{\mathrm{TOT}}(\vec{x})\, \mathrm{d}\vec{x}}{\sqrt{(\vec{r}-\vec{x})^T \overset{\leftrightarrow}{\epsilon}^{-1} (\vec{r}-\vec{x})}}. \tag{4}$$

This is true because of the linearity and space invariant properties of the original partial differential equation analyzed.

3. Eigenfield and inclusion concepts

We define a certain region of the space as an *inclusion* when the constitutive equation, in that zone, assumes the form $\vec{D} = \overleftrightarrow{\epsilon}(\vec{E} - \vec{E}^*)$, where $\vec{E}^*(\vec{r})$ is an assigned vector function of the position $\vec{r}$, which is named an *eigenfield*. We remark that the concept of inclusion is determined by the presence of a given eigenfield, which modifies the constitutive equation as discussed above, but it is not connected with the permittivity tensor $\overleftrightarrow{\epsilon}$, which remains homogeneous in the entire space. In this work, in order to indicate spatial variations of the permittivity tensor, we adopt the term inhomogeneity (see the next sections). The eigenfield, defined in some region of the space, acts as a sort of source of the electric field and its effects can be studied as follows. If the free charge distribution is absent in this region, we may use the Gauss equation $\vec{\nabla} \cdot \vec{D} = 0$, obtaining $\vec{\nabla} \cdot [\overleftrightarrow{\epsilon}(\vec{E} - \vec{E}^*)] = 0$ or, equivalently, $\vec{\nabla} \cdot [\overleftrightarrow{\epsilon}\vec{E}] = \vec{\nabla} \cdot [\overleftrightarrow{\epsilon}\vec{E}^*]$. Now, we can introduce the electric potential in the standard way, writing the relation $\vec{\nabla} \cdot [\overleftrightarrow{\epsilon}\vec{\nabla}V(\vec{r})] = -\vec{\nabla} \cdot [\overleftrightarrow{\epsilon}\vec{E}^*]$ or, similarly, the generalized Poisson equation $\vec{\nabla} \cdot [\overleftrightarrow{\epsilon}\vec{\nabla}V(\vec{r})] = -\rho^*$, where $\rho^* = \vec{\nabla} \cdot [\overleftrightarrow{\epsilon}\vec{E}^*]$ is the charge distribution equivalent to the eigenfield. In this context, the introduction of the concepts of eigenfield and inclusion has not a given direct physical meaning but it is a physical-mathematical expedient very useful to solve some problems in the electrostatics of the inhomogeneities, as we will show later on. So, we want to analyze the effects of the presence of a given inclusion (described by its eigenfield) on the electrical quantities. To begin with, we suppose that the eigenfield is defined in the whole three-dimensional space and, therefore, we may solve the generalized Poisson equation by means of the Green function introduced in the previous section,

$$V_{\mathrm{TOT}}(\vec{r}) = \frac{1}{4\pi\sqrt{\det\overleftrightarrow{\epsilon}}}\int_{\Re^3}\frac{\frac{\partial}{\partial x_l}[\epsilon_{lk}E_k^*(\vec{x})]\,\mathrm{d}\vec{x}}{\sqrt{(\vec{r}-\vec{x})^T\overleftrightarrow{\epsilon}^{-1}(\vec{r}-\vec{x})}}. \tag{5}$$

Now, we can use an integration by part, holding on for multiple integrals, which can be written as follows:

$$\int_{\Re^3}\vartheta(\vec{x})\frac{\partial\lambda(\vec{x})}{\partial x_l}\,\mathrm{d}\vec{x} = -\int_{\Re^3}\lambda(\vec{x})\frac{\partial\vartheta(\vec{x})}{\partial x_l}\,\mathrm{d}\vec{x} \tag{6}$$

where $\vartheta(\vec{x})$ and $\lambda(\vec{x})$ are two given functions with sufficiently regular behavior at infinity (this property is an immediate consequence of the Gauss-Ostrogradsky theorem). Application of this relation to equation (5) simply leads to

$$V_{\mathrm{TOT}}(\vec{r}) = -\frac{1}{4\pi\sqrt{\det\overleftrightarrow{\epsilon}}}\int_{\Re^3}\epsilon_{lk}E_k^*(\vec{x})\frac{\partial}{\partial x_l}\left(\frac{1}{\sqrt{(\vec{r}-\vec{x})^T\overleftrightarrow{\epsilon}^{-1}(\vec{r}-\vec{x})}}\right)\mathrm{d}\vec{x}. \tag{7}$$

Moreover, it easy to recognize the validity of the expression

$$\frac{\partial}{\partial x_l}\frac{1}{\sqrt{(\vec{r}-\vec{x})^T\overleftrightarrow{\epsilon}^{-1}(\vec{r}-\vec{x})}} = -\frac{\partial}{\partial r_l}\frac{1}{\sqrt{(\vec{r}-\vec{x})^T\overleftrightarrow{\epsilon}^{-1}(\vec{r}-\vec{x})}} \tag{8}$$

and therefore we can put equation (7) in the following form,

$$V_{\mathrm{TOT}}(\vec{r}) = \frac{1}{4\pi\sqrt{\det\overleftrightarrow{\epsilon}}}\int_{\Re^3}\epsilon_{lk}E_k^*(\vec{x})\frac{\partial}{\partial r_l}\left(\frac{1}{\sqrt{(\vec{r}-\vec{x})^T\overleftrightarrow{\epsilon}^{-1}(\vec{r}-\vec{x})}}\right)\mathrm{d}\vec{x}. \tag{9}$$

If the eigenfield $E_k^*(\vec{x})$ is constant in a limited region V of the space, we can say that we are dealing with a *uniform* or *homogeneous inclusion* V and the relative electric potential over the entire space becomes

$$V_{\mathrm{TOT}}(\vec{r}) = \frac{1}{4\pi\sqrt{\det \overleftrightarrow{\epsilon}}}\epsilon_{lk}E_k^*\frac{\partial}{\partial r_l}\int_V \frac{\mathrm{d}\vec{x}}{\sqrt{(\vec{r}-\vec{x})^T\overleftrightarrow{\epsilon}^{-1}(\vec{r}-\vec{x})}}. \tag{10}$$

We define the *anisotropic harmonic potential* of an arbitrary region V as

$$\psi_V(\vec{r}) = \int_V \frac{\mathrm{d}\vec{x}}{\sqrt{(\vec{r}-\vec{x})^T\overleftrightarrow{\epsilon}^{-1}(\vec{r}-\vec{x})}}. \tag{11}$$

So equation (10) can be written

$$V_{\mathrm{TOT}}(\vec{r}) = \frac{1}{4\pi\sqrt{\det \overleftrightarrow{\epsilon}}}\epsilon_{lk}E_k^*\frac{\partial\psi_V(\vec{r})}{\partial r_l}. \tag{12}$$

Moreover, we define the *isotropic harmonic potential* of a region Ω as

$$\Phi_\Omega(\vec{q}) = \int_\Omega \frac{\mathrm{d}\vec{y}}{\sqrt{(\vec{q}-\vec{y})^T(\vec{q}-\vec{y})}}. \tag{13}$$

In electrostatics and, more generally, in physics and mathematics many useful properties are well known for the isotropic harmonic potential [19, 30, 31]: in order to utilize these properties in our context, it could be interesting to obtain a relation between the anisotropic and the isotropic potentials defined in equations (11) and (13), respectively. To this aim, we undertake the following line of reasoning starting from equation (11) and recalling the diagonalization $\overleftrightarrow{\epsilon} = \overleftrightarrow{R}^T\overleftrightarrow{\Delta}\overleftrightarrow{R}$ (see appendix A) for the permittivity tensor

$$\begin{aligned}\psi_V(\vec{r}) &= \int_V \frac{\mathrm{d}\vec{x}}{\sqrt{(\vec{r}-\vec{x})^T\overleftrightarrow{R}^T\overleftrightarrow{\Delta}^{-1}\overleftrightarrow{R}(\vec{r}-\vec{x})}}\\ &= \int_V \frac{\mathrm{d}\vec{x}}{\sqrt{[\overleftrightarrow{\Delta}^{-1/2}\overleftrightarrow{R}(\vec{r}-\vec{x})]^T[\overleftrightarrow{\Delta}^{-1/2}\overleftrightarrow{R}(\vec{r}-\vec{x})]}}.\end{aligned} \tag{14}$$

By using the substitution $\vec{y} = \overleftrightarrow{\Delta}^{-1/2}\overleftrightarrow{R}\vec{x}$, which leads to $\mathrm{d}\vec{x} = \sqrt{\det \overleftrightarrow{\epsilon}}\,\mathrm{d}\vec{y}$, we obtain

$$\psi_V(\vec{r}) = \sqrt{\det \overleftrightarrow{\epsilon}}\int_{\overleftrightarrow{\Delta}^{-1/2}\overleftrightarrow{R}V}\frac{\mathrm{d}\vec{y}}{\|\overleftrightarrow{\Delta}^{-1/2}\overleftrightarrow{R}\vec{r}-\vec{y}\|} = \sqrt{\det \overleftrightarrow{\epsilon}}\,\Phi_{\overleftrightarrow{\Delta}^{-1/2}\overleftrightarrow{R}V}(\overleftrightarrow{\Delta}^{-1/2}\overleftrightarrow{R}\vec{r}) \tag{15}$$

where the symbol $\overleftrightarrow{\Delta}^{-1/2}\overleftrightarrow{R}V$ represents the deformation of the volume V under the effect of the linear operator $\overleftrightarrow{\Delta}^{-1/2}\overleftrightarrow{R}$. A very simple property of the isotropic harmonic potential is the following,

$$\Phi_\Omega(\overleftrightarrow{R}\vec{q}) = \Phi_{\overleftrightarrow{R}^T\Omega}(\vec{q}), \tag{16}$$

holding on for any orthogonal rotation matrix $\overleftrightarrow{R}$. This property can easily be verified as follows:

$$\Phi_\Omega(\overleftrightarrow{R}\vec{q}) = \int_\Omega\frac{\mathrm{d}\vec{y}}{\|\overleftrightarrow{R}\vec{q}-\vec{y}\|} = \int_{\overleftrightarrow{R}^T\Omega}\frac{\mathrm{d}\vec{v}}{\|\overleftrightarrow{R}\vec{q}-\overleftrightarrow{R}\vec{v}\|} = \int_{\overleftrightarrow{R}^T\Omega}\frac{\mathrm{d}\vec{v}}{\|\vec{q}-\vec{v}\|} = \Phi_{\overleftrightarrow{R}^T\Omega}(\vec{q}) \tag{17}$$

where we have used the substitution $\vec{y} = \overleftrightarrow{R}\vec{v}$ which leads to $\mathrm{d}\vec{y} = \mathrm{d}\vec{v}$ since the determinant of a rotation matrix is unitary (we have also considered $\|\overleftrightarrow{R}\vec{q} - \overleftrightarrow{R}\vec{v}\| = \|\vec{q}-\vec{v}\|$: a rotation matrix does not alter the length of a vector). Now, we can apply the property given in equation (16) to equation (15), obtaining

$$\begin{aligned}\psi_V(\vec{r}) &= \sqrt{\det \overleftrightarrow{\epsilon}}\,\Phi_{\overleftrightarrow{\Delta}^{-1/2}\overleftrightarrow{R}V}(\overleftrightarrow{\Delta}^{-1/2}\overleftrightarrow{R}\vec{r}) \\ &= \sqrt{\det \overleftrightarrow{\epsilon}}\,\Phi_{\overleftrightarrow{\Delta}^{-1/2}\overleftrightarrow{R}V}\big(\underbrace{\overleftrightarrow{R}\overleftrightarrow{R}^T}_{\overleftrightarrow{I}}\overleftrightarrow{\Delta}^{-1/2}\overleftrightarrow{R}\vec{r}\big) \\ &= \sqrt{\det \overleftrightarrow{\epsilon}}\,\Phi_{\overleftrightarrow{R}^T\overleftrightarrow{\Delta}^{-1/2}\overleftrightarrow{R}V}(\overleftrightarrow{R}^T\overleftrightarrow{\Delta}^{-1/2}\overleftrightarrow{R}\vec{r}) \\ &= \sqrt{\det \overleftrightarrow{\epsilon}}\,\Phi_{\sqrt{\overleftrightarrow{\epsilon}^{-1}}V}(\sqrt{\overleftrightarrow{\epsilon}^{-1}}\vec{r})\end{aligned} \tag{18}$$

where we have used again the diagonalization $\overleftrightarrow{\epsilon} = \overleftrightarrow{R}^T\overleftrightarrow{\Delta}\overleftrightarrow{R}$ for the symmetric permittivity tensor. The operation $\sqrt{\overleftrightarrow{\epsilon}}$ assumes a precise meaning by writing $\sqrt{\overleftrightarrow{\epsilon}} = \overleftrightarrow{R}^T\sqrt{\overleftrightarrow{\Delta}}\overleftrightarrow{R}$ where $\sqrt{\overleftrightarrow{\Delta}}$ is simply the tensor with the square roots of the diagonal elements of $\overleftrightarrow{\Delta}$ (of the principal permittivities); in fact, the product $\sqrt{\overleftrightarrow{\epsilon}}\sqrt{\overleftrightarrow{\epsilon}}$ can be simply calculated obtaining, as result, the permittivity tensor $\overleftrightarrow{\epsilon}$ itself. Similar considerations can be also applied to the square root of the inverse tensor $\overleftrightarrow{\epsilon}^{-1}$, which can be written as $\sqrt{\overleftrightarrow{\epsilon}^{-1}} = \overleftrightarrow{R}^T\sqrt{\overleftrightarrow{\Delta}^{-1}}\overleftrightarrow{R}$. However, equation (18) is the researched relationship between the anisotropic and the isotropic potentials.

Finally, by composing equations (12) and (18), we obtain the following general relation for the electric potential generated, in the whole anisotropic space, by a uniform eigenfield,

$$V_{\mathrm{TOT}}(\vec{r}) = \frac{1}{4\pi}\epsilon_{lk}E_k^*\frac{\partial}{\partial r_l}\Phi_{\sqrt{\overleftrightarrow{\epsilon}^{-1}}V}(\sqrt{\overleftrightarrow{\epsilon}^{-1}}\vec{r}), \tag{19}$$

written in terms of the isotropic harmonic potential defined in equation (13) (but related to the region $\sqrt{\overleftrightarrow{\epsilon}^{-1}}V$ and calculated in the argument $\sqrt{\overleftrightarrow{\epsilon}^{-1}}\vec{r}$). This result will play a crucial role in the development of the theory.

4. Ellipsoidal uniform inclusion

We wish to specialize the result given in equation (19) for an inclusion with an ellipsoidal shape. So, we assume that the region V is described by

$$V : \sum_{i=1}^{3}\frac{r_i^2}{a_i^2} \leqslant 1 \qquad \text{or} \qquad \vec{r}^T\overleftrightarrow{a}^{-2}\vec{r} \leqslant 1 \tag{20}$$

where a_1, a_2 and a_3 are the semi-axes of the ellipsoid aligned with the reference frame and we have defined a diagonal matrix $\overleftrightarrow{a}$ with the diagonal elements equal to a_1, a_2 and a_3. In order to develop the integral in equation (19) we may better characterize the region $\sqrt{\overleftrightarrow{\epsilon}^{-1}}V$. If a point $\vec{y}$ belongs to $\sqrt{\overleftrightarrow{\epsilon}^{-1}}V$ we can write $\vec{y} = \sqrt{\overleftrightarrow{\epsilon}^{-1}}\vec{r}$ where $\vec{r} \in V$; so by considering the inverse relation $\vec{r} = \sqrt{\overleftrightarrow{\epsilon}}\vec{y}$ and substituting it in equation (20), we obtain a useful description for $\sqrt{\overleftrightarrow{\epsilon}^{-1}}V$

$$\sqrt{\overleftrightarrow{\epsilon}^{-1}}V : \quad \vec{y}^T(\sqrt{\overleftrightarrow{\epsilon}}\overleftrightarrow{a}^{-2}\sqrt{\overleftrightarrow{\epsilon}})\vec{y} \leqslant 1. \tag{21}$$

So, the region $\sqrt{\overleftrightarrow{\epsilon}^{-1}}V$ is again ellipsoidal but the ellipsoid is not aligned to the axes of the reference frame and it is described by the tensor $\sqrt{\overleftrightarrow{\epsilon}}\overleftrightarrow{a}^{-2}\sqrt{\overleftrightarrow{\epsilon}}$. Therefore, such a tensor is not diagonal but symmetric, positive definite and always diagonalizable by means of a suitable rotation matrix $\overleftrightarrow{P}$

$$\sqrt{\overleftrightarrow{\epsilon}}\,\overleftrightarrow{a}^{-2}\sqrt{\overleftrightarrow{\epsilon}} = \overleftrightarrow{P}^T\,\overleftrightarrow{b}^{-2}\,\overleftrightarrow{P} \tag{22}$$

where $\overleftrightarrow{b}$ is a diagonal matrix containing the three semi-axes b_1, b_2 and b_3 of the rotated ellipsoid $\sqrt{\overleftrightarrow{\epsilon}^{-1}}V$; of course, these three values are the eigenvalues of the tensor $\sqrt{\overleftrightarrow{\epsilon}}\,\overleftrightarrow{a}^{-2}\sqrt{\overleftrightarrow{\epsilon}}$. Moreover, it is important to observe that the rotation matrix $\overleftrightarrow{P}$, which diagonalizes $\sqrt{\overleftrightarrow{\epsilon}}\,\overleftrightarrow{a}^{-2}\sqrt{\overleftrightarrow{\epsilon}}$, is different from the previously introduced rotation matrix $\overleftrightarrow{R}$, which instead diagonalizes the permittivity tensor $\overleftrightarrow{\epsilon}$. Anyway, a point $\vec{y}$ belongs to $\sqrt{\overleftrightarrow{\epsilon}^{-1}}V$ if $\vec{y}^T(\overleftrightarrow{P}^T\overleftrightarrow{b}^{-2}\overleftrightarrow{P})\vec{y} \leqslant 1$ or, similarly, if $(\overleftrightarrow{P}\vec{y})^T\overleftrightarrow{b}^{-2}(\overleftrightarrow{P}\vec{y}) \leqslant 1$; hence, we define a point $\vec{z} = \overleftrightarrow{P}\vec{y}$ belonging to the domain $V' = \overleftrightarrow{P}\sqrt{\overleftrightarrow{\epsilon}^{-1}}V$. So, the point $\vec{z}$ belongs to V' if and only if $\vec{z}^T\overleftrightarrow{b}^{-2}\vec{z} \leqslant 1$. Therefore, since $V' = \overleftrightarrow{P}\sqrt{\overleftrightarrow{\epsilon}^{-1}}V$, we can write $\sqrt{\overleftrightarrow{\epsilon}^{-1}}V = \overleftrightarrow{P}^T V'$, where V' is the ellipsoid $\vec{z}^T\overleftrightarrow{b}^{-2}\vec{z} \leqslant 1$ aligned with the reference frame under consideration. At the end of these considerations we may transform the isotropic potential appearing in equation (19) as follows:

$$\Phi_{\sqrt{\overleftrightarrow{\epsilon}^{-1}}V}(\sqrt{\overleftrightarrow{\epsilon}^{-1}}\vec{r}) = \Phi_{\overleftrightarrow{P}^T V'}(\sqrt{\overleftrightarrow{\epsilon}^{-1}}\vec{r}) = \Phi_{V'}(\overleftrightarrow{P}\sqrt{\overleftrightarrow{\epsilon}^{-1}}\vec{r}). \tag{23}$$

In the second equality we have used the property given in equation (16) and described in the previous section. Finally, the electric potential generated, in the whole anisotropic space, by a uniform ellipsoidal eigenfield is given by

$$V_{\mathrm{TOT}}(\vec{r}) = \frac{1}{4\pi}\epsilon_{lk}E_k^*\frac{\partial}{\partial r_l}\Phi_{V'}(\overleftrightarrow{P}\sqrt{\overleftrightarrow{\epsilon}^{-1}}\vec{r}). \tag{24}$$

We remember that $\overleftrightarrow{P}$ and $\overleftrightarrow{b}$ are defined by the diagonalization $\sqrt{\overleftrightarrow{\epsilon}}\,\overleftrightarrow{a}^{-2}\sqrt{\overleftrightarrow{\epsilon}} = \overleftrightarrow{P}^T\overleftrightarrow{b}^{-2}\overleftrightarrow{P}$ and the region V' is defined by all the points $\vec{z}$ satisfying $\vec{z}^T\overleftrightarrow{b}^{-2}\vec{z} \leqslant 1$. The final result given in equation (24) is the most important result in order to write the electric potential $V_{\mathrm{TOT}}(\vec{r})$ in closed form: in fact, now, the isotropic electric potential $\Phi_{V'}$ is written in terms of an ellipsoidal region aligned to the reference frame. In the following sections, we find out simple analytical expressions for the electric field both inside ($\vec{r} \in V$) and outside ($\vec{r} \in \Re^3\backslash V$) the ellipsoidal inclusion. Details of the two calculations can be found in appendices B and C, respectively.

4.1. Electric field inside the inclusion

The aim of this section is to find an explicit expression for the electric field generated inside the inclusion in terms of the uniform eigenfield defined above (we remember that $\vec{z} \in V'$ if and only if $\vec{r} \in V$). We need to develop equation (24) with a well-known important property of the isotropic harmonic potential, described below. The isotropic harmonic potential defined by

$$\Phi_{V'}(\vec{z}) = \int_{V'}\frac{\mathrm{d}\vec{p}}{\|\vec{z}-\vec{p}\|}, \tag{25}$$

where V' is the region defined by $\vec{z}^T\overleftrightarrow{b}^{-2}\vec{z} \leqslant 1$, can be represented by means of the following integral [19, 30, 31], which is a classical result in potential theory

$$\Phi_{V'}(\vec{z}) = \pi b_1 b_2 b_3\int_0^{+\infty}\frac{1-f(\vec{z},s)}{R(s)}\,\mathrm{d}s \tag{26}$$

for $\vec{z}$ belonging to V'. The functions $f(\vec{z}, s)$ and $R(s)$ are defined as follows:

$$f(\vec{z}, s) = \sum_{i=1}^{3} \frac{z_i^2}{b_i^2 + s} \tag{27}$$

$$R(s) = \sqrt{\left(b_1^2 + s\right)\left(b_2^2 + s\right)\left(b_3^2 + s\right)}. \tag{28}$$

Such an integral representation can be utilized in equation (24), as described in appendix B, in order to obtain the final result for the internal electric field in the simple form

$$\vec{E} = \left\{ \frac{\det(\overleftrightarrow{a})}{2} \int_0^{+\infty} \frac{(\overleftrightarrow{a}^2 + s\overleftrightarrow{\epsilon})^{-1}\overleftrightarrow{\epsilon}}{\sqrt{\det(\overleftrightarrow{a}^2 + s\overleftrightarrow{\epsilon})}} \,\mathrm{d}s \right\} \vec{E}^*. \tag{29}$$

4.2. Electric field outside the inclusion

Now, the aim is to find an explicit expression for the electric field generated outside the inclusion in terms of the uniform eigenfield (we remember that $\vec{z} \notin V'$ if and only if $\vec{r} \notin V$). Again, we need to develop equation (24) with another important property of the isotropic harmonic potential. It is defined in equation (25) and it can be represented by means of the following integral [19, 30, 31], which is another classical result in potential theory

$$\Phi_{V'}(\vec{z}) = \pi b_1 b_2 b_3 \int_{\eta(\vec{z})}^{+\infty} \frac{1 - f(\vec{z}, s)}{R(s)} \,\mathrm{d}s \tag{30}$$

in the region outside V' (i.e. $\vec{z} \in \Re^3 \backslash V'$). The functions $f(\vec{z}, s)$ and $R(s)$ are defined in equation (27) and the quantity $\eta(\vec{z})$ satisfies the relation $f(\vec{z}, \eta(\vec{z})) = 1$. Such an integral representation can be utilized in equation (24), as described in appendix C, in order to obtain the final result for the external electric field in the form

$$\begin{aligned} \vec{E}(\vec{r}) = {} & \frac{\det(\overleftrightarrow{a})}{2} \int_{\eta}^{+\infty} \frac{(\overleftrightarrow{a}^2 + s\overleftrightarrow{\epsilon})^{-1}\overleftrightarrow{\epsilon}}{\sqrt{\det(\overleftrightarrow{a}^2 + s\overleftrightarrow{\epsilon})}} \mathrm{d}s\, \vec{E}^* \\ & - \frac{\det(\overleftrightarrow{a})}{\sqrt{\det(\overleftrightarrow{a}^2 + \eta\overleftrightarrow{\epsilon})}} \frac{[(\overleftrightarrow{a}^2 + \eta\overleftrightarrow{\epsilon})^{-1}\vec{r}][\overleftrightarrow{\epsilon}(\overleftrightarrow{a}^2 + \eta\overleftrightarrow{\epsilon})^{-1}\vec{r}]^T}{[(\overleftrightarrow{a}^2 + \eta\overleftrightarrow{\epsilon})^{-1}\vec{r}]^T[\overleftrightarrow{\epsilon}(\overleftrightarrow{a}^2 + \eta\overleftrightarrow{\epsilon})^{-1}\vec{r}]} \vec{E}^* \end{aligned} \tag{31}$$

where the function $\eta(\vec{r})$ is implicitely defined by $\vec{r}^T(\overleftrightarrow{a}^2 + \eta\overleftrightarrow{\epsilon})^{-1}\vec{r} = 1$.

5. The electric Eshelby tensors

At this stage of the work we have completely solved the problem of a homogeneous ellipsoidal inclusion in an anisotropic environment. This solution can be summarized as follows. Both equations (29) for the internal field and (31) for the external one define a tensor which acts on the eigenfield to give the effective electric field in the corresponding region. By adopting a terminology taken from the elasticity theory we may call such tensors the internal electric Eshelby tensor $\overleftrightarrow{S}$ and the external electric Eshelby tensor $\overleftrightarrow{S}^{\infty}(\vec{r})$ [10, 11]. Their complete expressions are as follows:

$$\overleftrightarrow{S} = \frac{\det(\overleftrightarrow{a})}{2} \int_0^{+\infty} \frac{(\overleftrightarrow{a}^2 + s\overleftrightarrow{\epsilon})^{-1}\overleftrightarrow{\epsilon}}{\sqrt{\det(\overleftrightarrow{a}^2 + s\overleftrightarrow{\epsilon})}} \,\mathrm{d}s \tag{32}$$

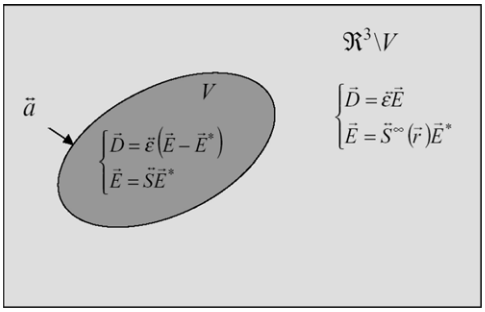

Figure 1. Scheme of an inclusion V with eigenfield $\vec{E}^*$ embedded in an anisotropic environment. The results in terms of the interior points and exterior point Eshelby tensors are also shown.

$$\overleftrightarrow{S}^\infty(\vec{r}) = \frac{\det(\overleftrightarrow{a})}{2}\int_\eta^{+\infty} \frac{(\overleftrightarrow{a}^2 + s\overleftrightarrow{\epsilon})^{-1}\overleftrightarrow{\epsilon}}{\sqrt{\det(\overleftrightarrow{a}^2 + s\overleftrightarrow{\epsilon})}}\,\mathrm{d}s - \frac{\det(\overleftrightarrow{a})}{\sqrt{\det(\overleftrightarrow{a}^2 + \eta\overleftrightarrow{\epsilon})}}\frac{[(\overleftrightarrow{a}^2 + \eta\overleftrightarrow{\epsilon})^{-1}\vec{r}][\overleftrightarrow{\epsilon}(\overleftrightarrow{a}^2 + \eta\overleftrightarrow{\epsilon})^{-1}\vec{r}]^T}{[(\overleftrightarrow{a}^2 + \eta\overleftrightarrow{\epsilon})^{-1}\vec{r}]^T[\overleftrightarrow{\epsilon}(\overleftrightarrow{a}^2 + \eta\overleftrightarrow{\epsilon})^{-1}\vec{r}]}. \tag{33}$$

Such definitions allow us to put the final equations for the electric field (see equations (29) and (31)) in a very simple form:

$$\begin{aligned} \vec{E} &= \overleftrightarrow{S}\vec{E}^* && \text{if} \quad \vec{r} \in V \\ \vec{E}(\vec{r}) &= \overleftrightarrow{S}^\infty(\vec{r})\vec{E}^* && \text{if} \quad \vec{r} \in \Re^3\backslash V. \end{aligned} \tag{34}$$

A summarizing scheme of the problem of an inclusion in an anisotropic environment can be found in figure 1, where the constitutive equations and the results in terms of the electric Eshelby tensors are reported in both internal and external regions.

6. Equivalence principle

In this section we show that the problem of an electrostatic inclusion, as stated in the first part of this work, is extremely useful to solve the problem of a given anisotropic inhomogeneity placed in an anisotropic matrix. We will show an equivalence principle that reduces the analysis of the behavior of an inhomogeneity to that of an inclusion. Let us start by considering an ellipsoidal inhomogeneity with permittivity tensor $\overleftrightarrow{\epsilon}_i$ embedded in an anisotropic environment with permittivity $\overleftrightarrow{\epsilon}$. We suppose that the whole structure is subjected to an external uniform electric field (remotely applied) $\vec{E}^\infty$ (of course we have $\vec{D}^\infty = \overleftrightarrow{\epsilon}\vec{E}^\infty$). We are searching for the perturbation to this uniform field induced by the presence of the inhomogeneity. The equivalence principle, which we are going to illustrate, has been summarized in figure 2. The actual presence of an inhomogeneity can be described by the superimposition of the effects generated by two different situations A and B. The first situation is very simple because it considers the effects of the remote field $\vec{E}^\infty$ in an homogeneous matrix without the inhomogeneity. In such a case, we simply observe that the electric displacement vector $\vec{D}^\infty = \overleftrightarrow{\epsilon}\vec{E}^\infty$ remains uniform in the entire space. Situation B corresponds to an inclusion

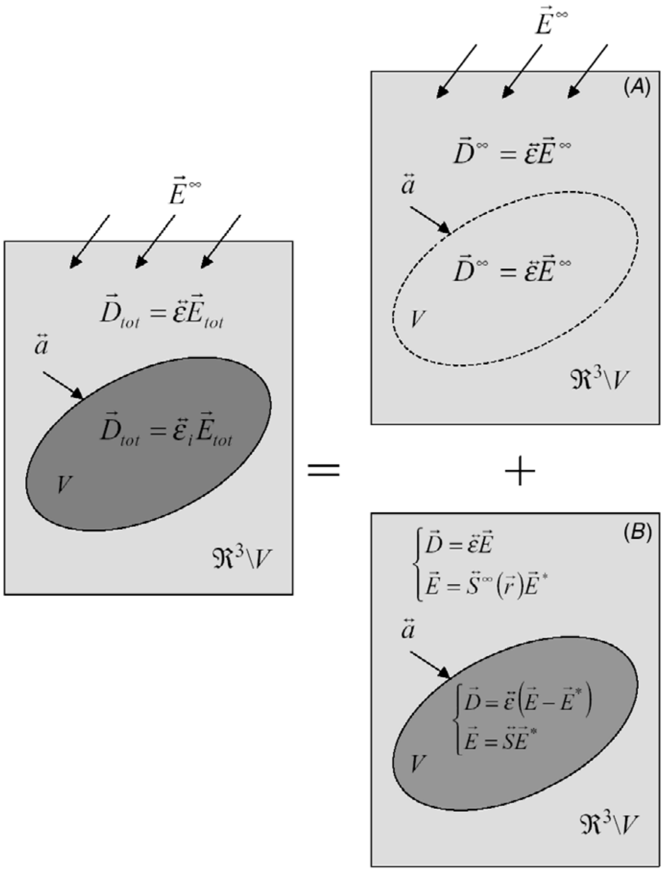

Figure 2. Scheme of the equivalence principle between the inhomogeneity problem and problem *A* (homogeneous medium) superimposed to problem *B* (inclusion as reported in figure 1).

scheme where the eigenfield $\vec{E}^*$ is still unknown and it can be determined by imposing the equivalence between the original problem and the superimposition A+B. We define as $\vec{D}_{\text{tot}}$ and $\vec{E}_{\text{tot}}$ the electric quantities in the initial inhomogeneity problem; as said above, the fields $\vec{D}^{\infty}$ and $\vec{E}^{\infty}$ correspond to the remote applied field and completely describe situation A; finally, problem B is described by the electric variables $\vec{D}$ and $\vec{E}$. Therefore, in any points of the space we have the superimposition $\vec{D}_{\text{tot}} = \vec{D}^{\infty} + \vec{D}$ and $\vec{E}_{\text{tot}} = \vec{E}^{\infty} + \vec{E}$. Hence, inside the ellipsoid we obtain

$$\begin{aligned} \overleftrightarrow{\epsilon}_i \vec{E}_{\text{tot}} &= \overleftrightarrow{\epsilon}\vec{E}^{\infty} + \overleftrightarrow{\epsilon}(\vec{E} - \vec{E}^*) \\ \vec{E}_{\text{tot}} &= \vec{E}^{\infty} + \vec{E}. \end{aligned} \tag{35}$$

These relationships, which must be verified in the internal region, allow us to calculate the exact value of the eigenfield $\vec{E}^*$ that assure the equivalence between the initial problem and the model A+B. Since $\vec{E} = \overleftrightarrow{S}\vec{E}^*$ for $\vec{r} \in V$, we may write

$$\begin{aligned} \overleftrightarrow{\epsilon}_i \vec{E}_{\text{tot}} &= \overleftrightarrow{\epsilon}\vec{E}^{\infty} + \overleftrightarrow{\epsilon}(\overleftrightarrow{S} - \overleftrightarrow{I})\vec{E}^* \\ \vec{E}_{\text{tot}} &= \vec{E}^{\infty} + \overleftrightarrow{S}\vec{E}^*. \end{aligned} \tag{36}$$

By substituting the second relation in the first one we have

$$\overleftrightarrow{\epsilon}_i(\vec{E}^\infty + \overleftrightarrow{S}\vec{E}^*) = \overleftrightarrow{\epsilon}\vec{E}^\infty + \overleftrightarrow{\epsilon}(\overleftrightarrow{S} - \overleftrightarrow{I})\vec{E}^*. \tag{37}$$

This is an equation in the eigenfield $\vec{E}^*$ that can be easily solved by obtaining

$$\vec{E}^* = [(\overleftrightarrow{I} - \overleftrightarrow{\epsilon}^{-1}\overleftrightarrow{\epsilon}_i)^{-1} - \overleftrightarrow{S}]^{-1}\vec{E}^\infty. \tag{38}$$

This is the value of the eigenfield that ensures the validity of the equivalence principle. Moreover, it is important to calculate the total electric field $\vec{E}_{\text{tot}}$ induced inside the inhomogeneity. From the second relation given in (36) we derive $\vec{E}^* = \overleftrightarrow{S}^{-1}(\vec{E}_{\text{tot}} - \vec{E}^\infty)$ and therefore, from the first one, we have

$$\overleftrightarrow{\epsilon}_i\vec{E}_{\text{tot}} = \overleftrightarrow{\epsilon}\vec{E}^\infty + \overleftrightarrow{\epsilon}(\overleftrightarrow{S} - \overleftrightarrow{I})\overleftrightarrow{S}^{-1}(\vec{E}_{\text{tot}} - \vec{E}^\infty). \tag{39}$$

This equation in the unknown $\vec{E}_{\text{tot}}$ can be solved with straightforward algebraic calculations, arriving at the solution

$$\vec{E}_{\text{tot}} = [\overleftrightarrow{I} - \overleftrightarrow{S}(\overleftrightarrow{I} - \overleftrightarrow{\epsilon}^{-1}\overleftrightarrow{\epsilon}_i)]^{-1}\vec{E}^\infty \tag{40}$$

when $\vec{r} \in V$. This is the internal electric field induced in the ellipsoid: this is a uniform vector field since all the quantities involved in equation (40) are constants.

Now, we can consider the external region: here the superimposition $\vec{E}_{\text{tot}} = \vec{E}^\infty + \vec{E}$ simply leads to the relation $\vec{E}_{\text{tot}}(\vec{r}) = \vec{E}^\infty + \overleftrightarrow{S}^\infty(\vec{r})\vec{E}^*$, which, by considering the eigenfield obtained in equation (38), leads immediately to the final result

$$\vec{E}_{\text{tot}}(\vec{r}) = \{\overleftrightarrow{I} + \overleftrightarrow{S}^\infty(\vec{r})[(\overleftrightarrow{I} - \overleftrightarrow{\epsilon}^{-1}\overleftrightarrow{\epsilon}_i)^{-1} - \overleftrightarrow{S}]^{-1}\}\vec{E}^\infty \tag{41}$$

when $\vec{r} \in \Re^3\backslash V$. The solutions given in equations (40) and (41), concerning the internal field and the external field respectively, become at last operative when they are coupled with the definitions of the Eshelby tensors, summarized in equations (32) and (33). Furthermore, we point out that these results have been written in terms of the matrix permittivity tensor $\overleftrightarrow{\epsilon}$, the inhomogeneity permittivity tensor $\overleftrightarrow{\epsilon}_i$, the ellipsoid tensor $\overleftrightarrow{a}$ (contained in the Eshelby tensors) and the externally applied field $\vec{E}^\infty$.

7. Generalization to nonlinear inhomogeneities

A nonlinear anisotropic (but homogenous) ellipsoid can be described from the electrical point of view by the constitutive equation

$$\vec{D} = \overleftrightarrow{\epsilon}_i(\vec{E})\vec{E} \tag{42}$$

where $\vec{D}$ is the electric displacement inside the particle, $\vec{E}$ is the electric field and the dielectric tensor function $\overleftrightarrow{\epsilon}_i(\vec{E})$ depends on the electric field $\vec{E}$. This functional relationship takes into account all the anisotropic and nonlinear possibilities for the electric behavior of the embedded particle. Let us now place this inhomogeneity in a linear (anisotropic) matrix characterized by the tensor permittivity $\overleftrightarrow{\epsilon}$ and let us calculate the field inside the ellipsoidal inclusion when a uniform external field $\vec{E}^\infty$ is applied to the system. If the particle were linear, in the dielectric sense, with constant permittivity $\overleftrightarrow{\epsilon}_i$, we would have, inside the ellipsoid, a uniform electric field $\vec{E}_{\text{tot}}$ given by equations (40) and (32), which lead to

$$\vec{E}_{\text{tot}} = \left[\overleftrightarrow{I} - \frac{\det(\overleftrightarrow{a})}{2}\int_0^{+\infty}\frac{(\overleftrightarrow{a}^2 + s\overleftrightarrow{\epsilon})^{-1}}{\sqrt{\det(\overleftrightarrow{a}^2 + s\overleftrightarrow{\epsilon})}}\,\mathrm{d}s(\overleftrightarrow{\epsilon} - \overleftrightarrow{\epsilon}_i)\right]^{-1}\vec{E}^\infty. \tag{43}$$

Conversely, if the sphere were electrically nonlinear, it is easy to prove that the internal field would satisfy the implicit equation

$$\vec{E}_{\rm tot} = \left[\overset{\leftrightarrow}{I} - \frac{\det(\overset{\leftrightarrow}{a})}{2} \int_0^{+\infty} \frac{(\overset{\leftrightarrow}{a}^2 + s\overset{\leftrightarrow}{\epsilon})^{-1}}{\sqrt{\det(\overset{\leftrightarrow}{a}^2 + s\overset{\leftrightarrow}{\epsilon})}} {\rm d}s[\vec{\epsilon} - \overset{\leftrightarrow}{\epsilon}_i(\vec{E}_{\rm tot})] \right]^{-1} \vec{E}^{\infty}. \tag{44}$$

This is true since the electric field $\vec{E}_{\rm tot}$ fulfilling equation (44) satisfies both Maxwell's laws and the boundary conditions at the inclusion surface as its linear counterpart (43) does when $\overset{\leftrightarrow}{\epsilon}_i = \overset{\leftrightarrow}{\epsilon}_i(\vec{E}_{\rm tot})$. In other words, if a solution of equation (44) exists, due to self-consistency, the problem is completely analogous to the linear one, provided that $\overset{\leftrightarrow}{\epsilon}_i = \overset{\leftrightarrow}{\epsilon}_i(\vec{E}_{\rm tot})$. Once equation (44) is solved for the internal field $\vec{E}_{\rm tot}$, the external field can be simply determined by equation (41), which continues to be valid also in this nonlinear case (since the nonlinearity is limited to the inhomogeneity).

8. Computational aspects of the external field

Both in the linear and nonlinear cases the external field is given by equation (41) where the external Eshelby tensor $\overset{\leftrightarrow}{S}^{\infty}(\vec{r})$ should be calculated from equation (33). In this relation appears the function η, which is implicitly defined by $\vec{r}^T(\overset{\leftrightarrow}{a}^2 + \eta\overset{\leftrightarrow}{\epsilon})^{-1}\vec{r} = 1$ (see equation (C.8)). Hence, when an arbitrary position vector $\vec{r}$ is given, we have to find the corresponding value of η that satisfies $\vec{r}^T(\overset{\leftrightarrow}{a}^2 + \eta\overset{\leftrightarrow}{\epsilon})^{-1}\vec{r} = 1$. This can be done with several computational approaches: here we present a simple iterative technique, which yields a simple implementation and a fast convergence. We start from equation (C.8), which can be easily rewritten in the form $\vec{r}^T\overset{\leftrightarrow}{a}^{-2}(\overset{\leftrightarrow}{I} + \eta\overset{\leftrightarrow}{\epsilon}\overset{\leftrightarrow}{a}^{-2})^{-1}\vec{r} = 1$. Now we can use the following general property holding for any tensor $\overset{\leftrightarrow}{A}$ and for any scalar quantity η,

$$(\overset{\leftrightarrow}{I} + \eta\overset{\leftrightarrow}{A})^{-1} = \overset{\leftrightarrow}{I} - \eta(\overset{\leftrightarrow}{A}^{-1} + \eta\overset{\leftrightarrow}{I})^{-1}. \tag{45}$$

Therefore, our implicit equation for η can be written as

$$\vec{r}^T\overset{\leftrightarrow}{a}^{-2}\{\overset{\leftrightarrow}{I} - \eta[(\overset{\leftrightarrow}{\epsilon}\overset{\leftrightarrow}{a}^{-2})^{-1} + \eta\overset{\leftrightarrow}{I}]^{-1}\}\vec{r} = 1 \tag{46}$$

With some rearrangements such a relation can be cast in the useful form

$$\eta = \frac{\vec{r}^T\overset{\leftrightarrow}{a}^{-2}\vec{r} - 1}{\vec{r}^T\overset{\leftrightarrow}{a}^{-2}[\overset{\leftrightarrow}{a}^2\overset{\leftrightarrow}{\epsilon}^{-1} + \eta\overset{\leftrightarrow}{I}]^{-1}\vec{r}}. \tag{47}$$

This equation is always in implicit form, but it can be used to define a recursive scheme as follows:

$$\begin{aligned} \eta_0(\vec{r}) &= 0 \\ \eta_{k+1}(\vec{r}) &= \frac{\vec{r}^T\overset{\leftrightarrow}{a}^{-2}\vec{r} - 1}{\vec{r}^T\overset{\leftrightarrow}{a}^{-2}[\overset{\leftrightarrow}{a}^2\overset{\leftrightarrow}{\epsilon}^{-1} + \eta_k(\vec{r})\overset{\leftrightarrow}{I}]^{-1}\vec{r}}. \end{aligned} \tag{48}$$

The initial value introduced in equation (48) corresponds to the value of η for $\vec{r}$ belonging to the ellipsoidal surface. An example of the behavior of this recursive scheme has been shown in figure 3, where we have assumed the following tensors in arbitrary units:

$$\overset{\leftrightarrow}{a} = \begin{bmatrix} 11 & 5 & 9 \\ 5 & 17 & 8 \\ 9 & 8 & 19 \end{bmatrix} \qquad \overset{\leftrightarrow}{\epsilon} = \begin{bmatrix} 7 & 2 & 3 \\ 2 & 11 & 8 \\ 3 & 8 & 9 \end{bmatrix}. \tag{49}$$

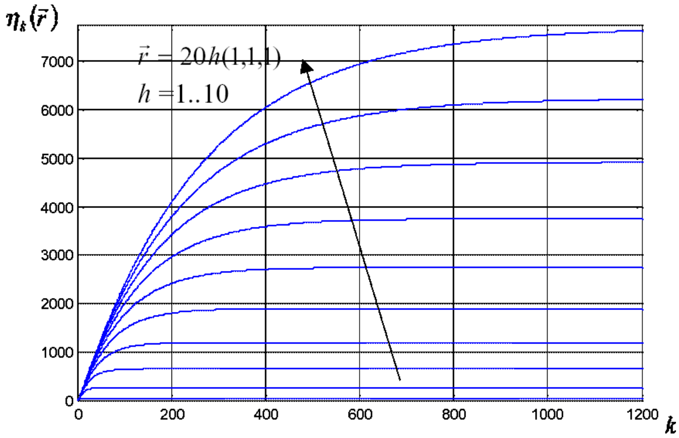

Figure 3. Convergence of the iterative algorithm defined in equation (48) for different values of the position vector $\vec{r}$ ($h = 1, \dots, 10$) and for several steps ($k = 1, \dots, 1200$). We have used the particular tensors defined in equation (49).

As one can verify, both such tensors are symmetric and positive definite, as required by their geometrical and physical meaning. We have plotted iterations for ten different values of $\vec{r}$ described by the simple law $\vec{r} = 20h(1, 1, 1)$, i.e., the vector $\vec{r}$ has modulus $20h\sqrt{3}$ and direction $(1, 1, 1)$ for h ranging from 1 to 10. We have performed iterations for 1200 steps and we have represented their behavior for each value of $\vec{r}$. We may observe that the convergence is slower for points with a greater distance from the origin of the reference frame. Nevertheless, the algorithm is always convergent in a practically acceptable number of iterations. Moreover, at the end of the procedure the relation $\vec{r}^T(\overset{\leftrightarrow}{a}^2 + \eta(\vec{r})\overset{\leftrightarrow}{\epsilon})^{-1}\vec{r} = 1$ is satisfied with great accuracy for any values of $\vec{r}$. Therefore, this simple method can be profitably utilized in numerical implementation of the theory described in this work.

9. Applications to anisotropic composite materials

In recent years there has been considerable interest in the linear and nonlinear response of anisotropic composite materials [32, 33]. The Maxwell–Garnett approximation [6] is one of the most widely used methods for calculating the equivalent dielectric properties of linear isotropic inhomogeneous materials. Here, the effective dielectric tensor of a dispersion of anisotropic inclusions embedded in an anisotropic host has been calculated using a generalization of the Maxwell–Garnett approximation. It involves an exact evaluation of the uniform electric field induced inside a single ellipsoidal inclusion and the evaluation of electrical quantities, averaged over the mixture volume. This approach has been extensively used for studying the properties of two-component mixtures in which both the host and the inclusions are isotropic [6, 7]. Also anisotropic spheres in an isotropic matrix have been largely studied by means of this method [34–36]. So, in order to generalize these situations we consider a population of anisotropic parallel ellipsoids ($\overset{\leftrightarrow}{\epsilon}_i$) randomly embedded in an anisotropic matrix ($\overset{\leftrightarrow}{\epsilon}$). The random character of the system is limited to the positions of the inhomogeneities and not to their orientations. We define the volume fraction of the embedded phase as c. The dilute limit is assumed ($c \ll 1$), so that each inclusion basically 'feels' only the uniform, externally applied, electric field. Therefore, the average value of the electric field over the whole structure is given by

$$\begin{aligned}\langle\vec{E}\rangle &= c[\overleftrightarrow{I} - \overleftrightarrow{S}(\overleftrightarrow{I} - \overleftrightarrow{\epsilon}^{-1}\overleftrightarrow{\epsilon}_i)]^{-1}\vec{E}^{\infty} + (1-c)\vec{E}^{\infty} \\ &= \{c[\overleftrightarrow{I} - \overleftrightarrow{S}(\overleftrightarrow{I} - \overleftrightarrow{\epsilon}^{-1}\overleftrightarrow{\epsilon}_i)]^{-1} + (1-c)\overleftrightarrow{I}\}\vec{E}^{\infty}\end{aligned} \tag{50}$$

where we have used equation (40) for the internal uniform field induced in each ellipsoid. The relation between the average electric field in the heterogeneous system and the external applied one is, therefore,

$$\vec{E}^{\infty} = \{c[\overleftrightarrow{I} - \overleftrightarrow{S}(\overleftrightarrow{I} - \overleftrightarrow{\epsilon}^{-1}\overleftrightarrow{\epsilon}_i)]^{-1} + (1-c)\overleftrightarrow{I}\}^{-1}\langle\vec{E}\rangle. \tag{51}$$

At this point we may evaluate the average value of the electric displacement in the dispersion of ellipsoids; we define V as the total volume of the mixture, V_e as the total volume of the embedded ellipsoids and V_o as the volume of the remaining space among the inclusions (so that $V = V_e \cup V_o$). So, the average value of $\vec{D}(\vec{r}) = \overleftrightarrow{\epsilon}(\vec{r})\vec{E}(\vec{r})$ follows:

$$\langle\vec{D}\rangle = \frac{1}{V}\int_V \overleftrightarrow{\epsilon}(\vec{r})\vec{E}(\vec{r})\,\mathrm{d}\vec{r} = \frac{1}{V}\overleftrightarrow{\epsilon}\int_{V_o}\vec{E}(\vec{r})\,\mathrm{d}\vec{r} + \frac{1}{V}\overleftrightarrow{\epsilon}_i\int_{V_e}\vec{E}(\vec{r})\,\mathrm{d}\vec{r} \tag{52}$$

or, equivalently

$$\begin{aligned}\langle\vec{D}\rangle &= \frac{1}{V}\left[\overleftrightarrow{\epsilon}\int_{V_o}\vec{E}\,\mathrm{d}\vec{r} + \overleftrightarrow{\epsilon}\int_{V_e}\vec{E}\,\mathrm{d}\vec{r} + \overleftrightarrow{\epsilon}_i\int_{V_e}\vec{E}\,\mathrm{d}\vec{r} - \overleftrightarrow{\epsilon}\int_{V_e}\vec{E}\,\mathrm{d}\vec{r}\right] \\ &= \overleftrightarrow{\epsilon}\langle\vec{E}\rangle + c(\overleftrightarrow{\epsilon}_i - \overleftrightarrow{\epsilon})\langle\vec{E}_{\mathrm{tot}}\rangle \\ &= \overleftrightarrow{\epsilon}\langle\vec{E}\rangle + c(\overleftrightarrow{\epsilon}_i - \overleftrightarrow{\epsilon})[\overleftrightarrow{I} - \overleftrightarrow{S}(\overleftrightarrow{I} - \overleftrightarrow{\epsilon}^{-1}\overleftrightarrow{\epsilon}_i)]^{-1}\vec{E}^{\infty}\end{aligned} \tag{53}$$

where $\langle\vec{E}_{\mathrm{tot}}\rangle$ is the average value of the electric field induced inside the ellipsoids; it can be considered uniform and therefore we obtain the final result given in equation (53) by taking into account (40). Now, by substituting equation (51) in (53) we simply obtain

$$\begin{aligned}\langle\vec{D}\rangle &= \overleftrightarrow{\epsilon}\langle\vec{E}\rangle + c(\overleftrightarrow{\epsilon}_i - \overleftrightarrow{\epsilon})[\overleftrightarrow{I} - \overleftrightarrow{S}(\overleftrightarrow{I} - \overleftrightarrow{\epsilon}^{-1}\overleftrightarrow{\epsilon}_i)]^{-1} \\ &\quad\times\{c[\overleftrightarrow{I} - \overleftrightarrow{S}(\overleftrightarrow{I} - \overleftrightarrow{\epsilon}^{-1}\overleftrightarrow{\epsilon}_i)]^{-1} + (1-c)\overleftrightarrow{I}\}^{-1}\langle\vec{E}\rangle \\ &= \overleftrightarrow{\epsilon}\langle\vec{E}\rangle + c(\overleftrightarrow{\epsilon}_i - \overleftrightarrow{\epsilon})\{c\overleftrightarrow{I} + (1-c)[\overleftrightarrow{I} - \overleftrightarrow{S}(\overleftrightarrow{I} - \overleftrightarrow{\epsilon}^{-1}\overleftrightarrow{\epsilon}_i)]\}^{-1}\langle\vec{E}\rangle.\end{aligned} \tag{54}$$

Then, we define $\overleftrightarrow{\epsilon}_{\mathrm{eff}}$ as the effective permittivity tensor of the whole mixture by means of the relation $\langle\vec{D}\rangle = \overleftrightarrow{\epsilon}_{\mathrm{eff}}\langle\vec{E}\rangle$. Drawing a comparison between this definition and equation (54) we may find a complete expression, which allows us to estimate the effective permittivity tensor

$$\overleftrightarrow{\epsilon}_{\mathrm{eff}} = \overleftrightarrow{\epsilon} + c(\overleftrightarrow{\epsilon}_i - \overleftrightarrow{\epsilon})\{\overleftrightarrow{I} + (1-c)\overleftrightarrow{S}\overleftrightarrow{\epsilon}^{-1}(\overleftrightarrow{\epsilon}_i - \overleftrightarrow{\epsilon})\}^{-1}. \tag{55}$$

By recalling the definition of the internal electric Eshelby tensor we obtain the final result as

$$\overleftrightarrow{\epsilon}_{\mathrm{eff}} = \overleftrightarrow{\epsilon} + c(\overleftrightarrow{\epsilon}_i - \overleftrightarrow{\epsilon})\left\{\overleftrightarrow{I} + (1-c)\frac{\det(\overleftrightarrow{a})}{2}\int_0^{+\infty}\frac{(\overleftrightarrow{a}^2 + s\overleftrightarrow{\epsilon})^{-1}}{\sqrt{\det(\overleftrightarrow{a}^2 + s\overleftrightarrow{\epsilon})}}\,\mathrm{d}s(\overleftrightarrow{\epsilon}_i - \overleftrightarrow{\epsilon})\right\}^{-1}. \tag{56}$$

This relation is valid for low values of the volume fraction c (is exact for $c = 0$). Nevertheless, quite surprisingly, it furnishes exact results also for the extreme case $c = 1$. Anyway, the results cannot be considered correct for large values of the volume fraction. In order to generalize this approach to higher values of the volume fraction we may adopt the differential scheme [4, 6]. We indicate with $\overleftrightarrow{\epsilon}_{\mathrm{diff}}$ the estimate of the dielectric tensor obtained with this method. As described in several works [4, 6], the incremental procedure leads to the following differential equation for the effective properties:

$$\frac{\mathrm{d}\overset{\leftrightarrow}{\epsilon}_{\mathrm{diff}}}{\mathrm{d}c} = \frac{1}{1-c}\left.\frac{\partial\overset{\leftrightarrow}{\epsilon}_{\mathrm{eff}}}{\partial c}\right|_{c=0,\overset{\leftrightarrow}{\epsilon}=\overset{\leftrightarrow}{\epsilon}_{\mathrm{diff}}} \tag{57}$$

where the initial condition is $\overset{\leftrightarrow}{\epsilon}_{\mathrm{diff}}(c=0) = \overset{\leftrightarrow}{\epsilon}$ and $\overset{\leftrightarrow}{\epsilon}_{\mathrm{eff}}$ is given by the dilute result shown in equation (56). Performing the operations indicated in equation (57), we simply find the explicit form of the effective differential method,

$$\frac{\mathrm{d}\overset{\leftrightarrow}{\epsilon}_{\mathrm{diff}}}{\mathrm{d}c} = \frac{1}{1-c}(\overset{\leftrightarrow}{\epsilon}_i - \overset{\leftrightarrow}{\epsilon}_{\mathrm{diff}})\left\{\overset{\leftrightarrow}{I} + \frac{\det(\overset{\leftrightarrow}{a})}{2}\int_0^{+\infty}\frac{(\overset{\leftrightarrow}{a}^2 + s\overset{\leftrightarrow}{\epsilon}_{\mathrm{diff}})^{-1}}{\sqrt{\det(\overset{\leftrightarrow}{a}^2 + s\overset{\leftrightarrow}{\epsilon}_{\mathrm{diff}})}}\,\mathrm{d}s(\overset{\leftrightarrow}{\epsilon}_i - \overset{\leftrightarrow}{\epsilon}_{\mathrm{diff}})\right\}^{-1} \tag{58}$$

with $\overset{\leftrightarrow}{\epsilon}_{\mathrm{diff}}(c=0) = \overset{\leftrightarrow}{\epsilon}$. We summarize the various quantities involved in the previous equations: $\overset{\leftrightarrow}{\epsilon}$ is the tensor permittivity of the matrix, $\overset{\leftrightarrow}{\epsilon}_i$ is the tensor permittivity of the parallel ellipsoids, $\overset{\leftrightarrow}{a}$ its tensor of the semi-axes of the ellipsoid and the solution $\overset{\leftrightarrow}{\epsilon}_{\mathrm{diff}}$ is the effective dielectric tensor. It is interesting to note that the components of the effective tensor are strongly analytically coupled in the system of differential equations appearing in equation (58). In order to show a simple example we consider the case with the tensors $\overset{\leftrightarrow}{\epsilon}$, $\overset{\leftrightarrow}{\epsilon}_i$, $\overset{\leftrightarrow}{a}$ and $\overset{\leftrightarrow}{\epsilon}_{\mathrm{diff}}$ contemporaneously diagonal in the same reference frame. This means that the geometrical principal axes of the ellipsoids are parallel to the optical principal axes of the ellipsoid and the matrix. Moreover, we assume an isotropic material for the inhomogeneities, i.e. $\overset{\leftrightarrow}{\epsilon}_i = \epsilon_i\overset{\leftrightarrow}{I}$ (ϵ_i takes the role of scalar permittivity for the ellipsoids). The external medium ($\overset{\leftrightarrow}{\epsilon}$) is instead considered anisotropic with three different principal permittivities ($\epsilon_{10}, \epsilon_{20}, \epsilon_{30}$). Similarly, the effective dielectric tensor $\overset{\leftrightarrow}{\epsilon}_{\mathrm{diff}}$ has the three principal permittivities ϵ_1, ϵ_2 and ϵ_3. Summing up, equation (58) corresponds to the following system:

$$\frac{\mathrm{d}\epsilon_k}{\mathrm{d}c} = \frac{1}{1-c}(\epsilon_i - \epsilon_k)\left\{1 + \frac{a_1a_2a_3}{2}\int_0^{+\infty}\frac{\mathrm{d}s\,(\epsilon_i - \epsilon_k)}{\left(a_k^2 + s\epsilon_k\right)\sqrt{\left(a_1^2 + s\epsilon_1\right)\left(a_2^2 + s\epsilon_2\right)\left(a_3^2 + s\epsilon_3\right)}}\right\}^{-1}$$
$$\epsilon_k(c=0) = \epsilon_{k0}. \tag{59}$$

Now, in order to obtain a simpler model, we consider, as a particular case, elliptic cylinders aligned with the x_3-axis; it means that we perform the limit $a_3 \to \infty$ in equation (59). The result is the following:

$$\begin{aligned}
\frac{\mathrm{d}\epsilon_1}{\mathrm{d}c} &= \frac{1}{1-c}(\epsilon_i - \epsilon_1)\left\{1 + \frac{a_1a_2}{2}\int_0^{+\infty}\frac{\mathrm{d}s(\epsilon_i - \epsilon_1)}{\left(a_1^2 + s\epsilon_1\right)^{3/2}\left(a_2^2 + s\epsilon_2\right)^{1/2}}\right\}^{-1}\\
\frac{\mathrm{d}\epsilon_2}{\mathrm{d}c} &= \frac{1}{1-c}(\epsilon_i - \epsilon_2)\left\{1 + \frac{a_1a_2}{2}\int_0^{+\infty}\frac{\mathrm{d}s(\epsilon_i - \epsilon_2)}{\left(a_1^2 + s\epsilon_1\right)^{1/2}\left(a_2^2 + s\epsilon_2\right)^{3/2}}\right\}^{-1}\\
\frac{\mathrm{d}\epsilon_3}{\mathrm{d}c} &= \frac{1}{1-c}(\epsilon_i - \epsilon_3)
\end{aligned} \tag{60}$$

where the initial conditions $\epsilon_1(c=0) = \epsilon_{10}, \epsilon_2(c=0) = \epsilon_{20}, \epsilon_3(c=0) = \epsilon_{30}$ must be considered. The integrals appearing in equation (60) can be solved in closed form by using the following result (that can be simply verified by means of the substitution $x = \sqrt{(a+bs)/(c+\mathrm{d}s)}$)

$$\frac{1}{2}\int_0^{+\infty}\frac{\mathrm{d}s}{(a+bs)^{3/2}(c+\mathrm{d}s)^{1/2}} = \frac{1}{\sqrt{ab}(\sqrt{ad}+\sqrt{bc})} \tag{61}$$

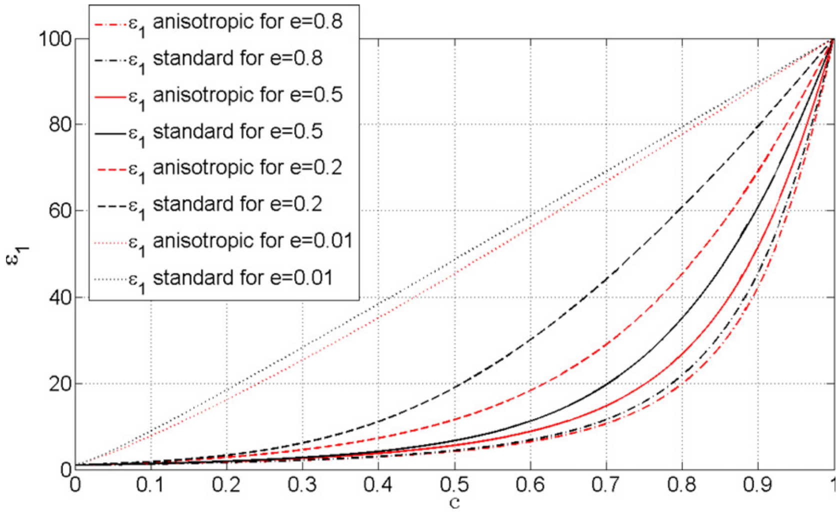

Figure 4. Comparison among the results for ϵ_1 obtained with the anisotropic differential system (62) and the standard differential method (63). The solutions of equations (62) are placed under the corresponding solutions (63).

which is correct for arbitrary positive numbers a, b, c and d. So, by assuming $\epsilon_{10} = \epsilon_{20} = \epsilon_{30} = \epsilon_m$ (scalar permittivity of the matrix) and by defining the aspect ratio $e = a_2/a_1$, we finally obtain

$$\begin{aligned}
\frac{\mathrm{d}\epsilon_1}{\mathrm{d}c} &= \frac{\epsilon_i - \epsilon_1}{1-c}\frac{\epsilon_1 e + \sqrt{\epsilon_1\epsilon_2}}{\epsilon_i e + \sqrt{\epsilon_1\epsilon_2}} \\
\frac{\mathrm{d}\epsilon_2}{\mathrm{d}c} &= \frac{\epsilon_i - \epsilon_2}{1-c}\frac{\epsilon_2 + e\sqrt{\epsilon_1\epsilon_2}}{\epsilon_i + e\sqrt{\epsilon_1\epsilon_2}} \\
\frac{\mathrm{d}\epsilon_3}{\mathrm{d}c} &= \frac{\epsilon_i - \epsilon_3}{1-c}
\end{aligned} \tag{62}$$

with $\epsilon_1\,(c=0) = \epsilon_2\,(c=0) = \epsilon_3\,(c=0) = \epsilon_m$. The solution of such a differential system can be compared with the standard solution obtained by means of the differential method applied without taking into account the anisotropic character of the system. These simpler and approximate solutions can be found in the literature [6] and they are reported below for completeness:

$$\begin{aligned}
1 - c &= \frac{\epsilon_i - \epsilon_1}{\epsilon_i - \epsilon_m}\left(\frac{\epsilon_i}{\epsilon_1}\right)^{\frac{e}{e+1}} \\
1 - c &= \frac{\epsilon_i - \epsilon_2}{\epsilon_i - \epsilon_m}\left(\frac{\epsilon_i}{\epsilon_2}\right)^{\frac{1}{e+1}} \\
\epsilon_3 &= c\epsilon_i + (1-c)\,\epsilon_m.
\end{aligned} \tag{63}$$

The solutions concerning the x_3 direction are perfectly coincident in both approaches and, therefore, we do not draw further comparison. Indeed, this relation is an exact result describing a parallel connection of capacitors (the interfaces are aligned with the electrical field). On the other hand, the first two relations in (62) and (63) require more refined checks. We have numerically solved the differential system formed by the first two equations (62) and the corresponding irrational equations (63). In figures 4 and 5 we have shown the results

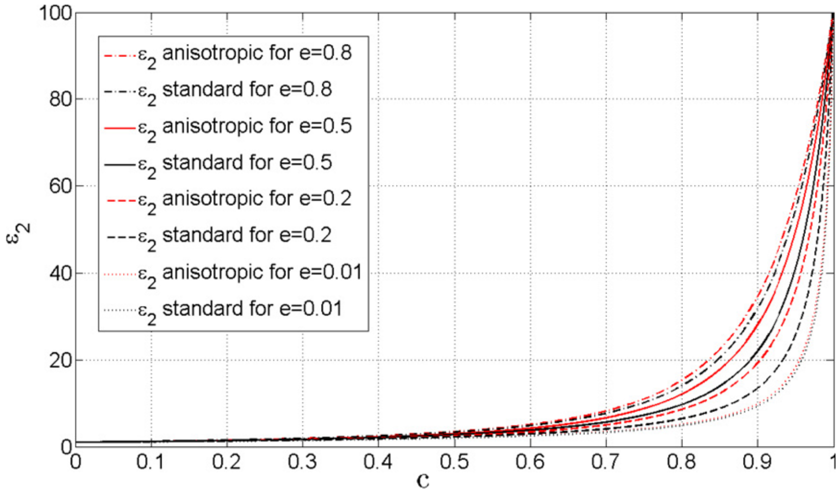

Figure 5. Comparison among the results for ϵ_2 obtained with the anisotropic differential system (62) and the standard differential method (63). The solutions of equations (62) are placed above the corresponding solutions of (63).

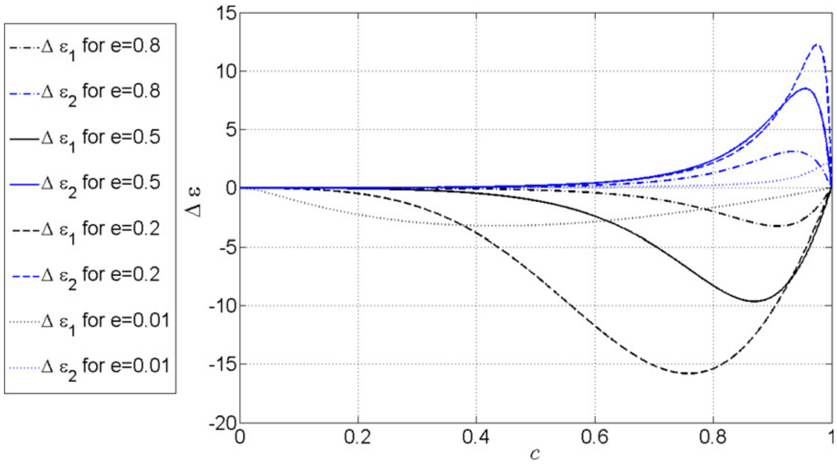

Figure 6. The variations $\Delta\epsilon$ (values obtained with equation (62) minus values obtained with equation (63)) are plotted versus the volume fraction c of the composite material. One can see that the relative error can assume values up to 20%.

in terms of the volume fraction c, by assuming $\epsilon_i = 100$ and $\epsilon_m = 1$ (in arbitrary units). We have drawn a comparison for different values of the aspect ratio of the elliptic cylinders ($e = 0.01$, 0.2, 0.5 and 0.8). Being $e < 1$, we have explored the situation wherein the axis along x_2 is shorter than the axis along x_1. Firstly, it is interesting to observe that the estimation of ϵ_1 with the improved (anisotropic) differential method (62) is less than the estimation obtained with the approximated differential method (63); on the other hand, the estimation of ϵ_2 with the anisotropic differential method (62) is greater than the estimation obtained with the standard differential method (63). This fact can be easily seen in figure 4 for ϵ_1, where the solutions of equation (62) are placed under the corresponding solutions of equation (63), and in figure 5 for ϵ_2, where the solutions of equation (62) are placed above the corresponding solutions of equation (63). The extent of the differences between the two approaches can be found in figure 6, where the variations $\Delta\epsilon$ (values obtained with equation (62) minus values obtained with equation (63)) are plotted versus the volume fraction c of the composite material. One can see that the relative error can assume values up to 20%.

10. Conclusions

We have introduced a methodology to cope with the problem of the electric behavior of an anisotropic inhomogeneity embedded in an anisotropic host matrix (environment). This problem has been solved by drawing a comparison with a similar one, placed within the elasticity theory. Therefore, we have developed a general theory (similar to the Eshelby one) describing the electric quantities both inside and outside the particle in the case of arbitrary anisotropic behavior of the involved materials. We have also described a generalization of the case of an arbitrarily nonlinear and anisotropic inhomogeneity. Finally, we have investigated the consequence of these achievements on the characterization of composite materials by means of the differential effective medium theory.

Appendix A. Anisotropic Green function

The differential problem stated in equation (1) or (2) can be straightforwardly solved by means of the three-dimensional Fourier transform, which converts the vector $\vec{r}$ to the vector $\vec{\Omega}$. As well known, the differential operators must be transformed via the rule $\partial/\partial x_k \to \mathrm{i}\Omega_k$, simply obtaining this result for the electrical potential in the transformed domain:

$$\tilde{V}(\vec{\Omega}) = \frac{Q}{\epsilon_{kl}\Omega_k\Omega_l} = \frac{Q}{\vec{\Omega}^T\overleftrightarrow{\epsilon}\vec{\Omega}}. \tag{A.1}$$

It is worth saying that any kind of anisotropy can be modeled through the procedure here presented, including uniaxial, biaxial and also gyrotropic media, in which the permittivity tensor contains an antisymmetric part. However, in the following, we consider a symmetric permittivity tensor in order to exploit diagonalization by means of a suitable orthogonal matrix $\overleftrightarrow{R}$; this is done for the sake of simplicity in the exposition and it does not restrict the generality of the present approach. Therefore, we may assume the diagonalization $\overleftrightarrow{\epsilon} = \overleftrightarrow{R}^T\overleftrightarrow{\Delta}\overleftrightarrow{R}$ where $\overleftrightarrow{\Delta}$ is a diagonal matrix with the principal permittivities of the medium (positive numbers). The electric potential can be written in the original spatial domain as follows:

$$V(\vec{r}) = \frac{1}{(2\pi)^3}\int_{\Re^3}\tilde{V}(\vec{\Omega})\exp(\mathrm{i}\vec{\Omega}^T\vec{r})\,\mathrm{d}\vec{\Omega} = \frac{Q}{(2\pi)^3}\int_{\Re^3}\frac{\exp(\mathrm{i}\vec{\Omega}^T\vec{r})\,\mathrm{d}\vec{\Omega}}{(\sqrt{\overleftrightarrow{\Delta}}\overleftrightarrow{R}\vec{\Omega})^T(\sqrt{\overleftrightarrow{\Delta}}\overleftrightarrow{R}\vec{\Omega})}. \tag{A.2}$$

Here $\sqrt{\overleftrightarrow{\Delta}}$ represents the diagonal matrix with the three square roots of the principal permittivities. The last integral can be simply handled with the substitution $\vec{y} = \sqrt{\overleftrightarrow{\Delta}}\overleftrightarrow{R}\vec{\Omega}$, which leads to the differential relation $\mathrm{d}\vec{y} = \sqrt{\det\overleftrightarrow{\Delta}}\,\mathrm{d}\vec{\Omega} = \sqrt{\det\overleftrightarrow{\epsilon}}\,\mathrm{d}\vec{\Omega}$ ($\det\overleftrightarrow{\Delta} = \det\overleftrightarrow{\epsilon}$ since $\overleftrightarrow{\Delta}$ and $\overleftrightarrow{\epsilon}$ are equivalent matrices). Thus, we have

$$V(\vec{r}) = \frac{Q}{(2\pi)^3\sqrt{\det\overleftrightarrow{\epsilon}}}\int_{\Re^3}\frac{\exp(\mathrm{i}\vec{r}^T\overleftrightarrow{R}^T\sqrt{\overleftrightarrow{\Delta}}^{-1}\vec{y})\,\mathrm{d}\vec{y}}{(\vec{y})^T(\vec{y})}. \tag{A.3}$$

Now, we may remember the well-known transform couple $1/\|\vec{r}\| \to 4\pi/\|\vec{\Omega}\|^2$ (the norm symbol $\|\vec{v}\|$ means $\sqrt{\vec{v}^T\vec{v}}$, as usual), which helps us to solve the integral in equation (A.3) as follows:

$$V(\vec{r}) = \frac{Q}{4\pi\sqrt{\det\overleftrightarrow{\epsilon}}}\frac{1}{\|\sqrt{\overleftrightarrow{\Delta}}^{-1}\overleftrightarrow{R}\vec{r}\|} = \frac{Q}{4\pi\sqrt{\det\overleftrightarrow{\epsilon}\,[\vec{r}^T\overleftrightarrow{\epsilon}^{-1}\vec{r}]}}. \tag{A.4}$$

We have described a complete proof for the sake of completeness and because the Green function is the fundamental starting point for this work; equation (A.4) can be also verified with other methods, as reported, e.g., in the Landau textbook [16].

Appendix B. Calculation of the internal field

Determination of the internal electric potential can be made through the combination of equation (24) with (26):

$$V_{\mathrm{TOT}}(\vec{r}) = \frac{1}{4\pi}\epsilon_{lk}E_k^*\frac{\partial}{\partial r_l}\left[\pi b_1b_2b_3\int_0^{+\infty}\frac{1-f(\vec{z},s)}{R(s)}\,\mathrm{d}s\right] \tag{B.1}$$

where $\vec{z} = \overleftrightarrow{P}\sqrt{\overleftrightarrow{\epsilon}}^{-1}\vec{r}$ is implicitely assumed. The electric field can be calculated differentiating the previous relation with respect to the position vector $\vec{r}$. In components we obtain

$$\begin{aligned} E_j(\vec{r}) &= \frac{\epsilon_{lk}E_k^*}{4}b_1b_2b_3\frac{\partial}{\partial r_j}\frac{\partial}{\partial r_l}\left[\int_0^{+\infty}\frac{f(\vec{z},s)}{R(s)}\,\mathrm{d}s\right] \\ &= \frac{\epsilon_{lk}E_k^*}{4}b_1b_2b_3\int_0^{+\infty}\sum_{i=1}^{3}\frac{1}{b_i^2+s}\left[\frac{\partial}{\partial r_j}\frac{\partial}{\partial r_l}z_i^2\right]\frac{\mathrm{d}s}{R(s)} \end{aligned} \tag{B.2}$$

where $\vec{z} = \overleftrightarrow{P}\sqrt{\overleftrightarrow{\epsilon}}^{-1}\vec{r}$ is again implicitely assumed. The term in brackets can be developed in this way (the index i is not summed):

$$\begin{aligned} \frac{\partial}{\partial r_j}\frac{\partial}{\partial r_l}\left(z_i^2\right)_{\vec{z}=\overleftrightarrow{P}\sqrt{\overleftrightarrow{\epsilon}}^{-1}\vec{r}} &= \frac{\partial}{\partial r_j}\frac{\partial}{\partial r_l}[P_{iq}(\sqrt{\overleftrightarrow{\epsilon}}^{-1})_{qs}r_s][P_{ip}(\sqrt{\overleftrightarrow{\epsilon}}^{-1})_{pr}r_r] \\ &= P_{iq}P_{ip}(\sqrt{\overleftrightarrow{\epsilon}}^{-1})_{qs}(\sqrt{\overleftrightarrow{\epsilon}}^{-1})_{pr}(\delta_{jr}\delta_{ls}+\delta_{rl}\delta_{js}) \\ &= P_{iq}P_{ip}(\sqrt{\overleftrightarrow{\epsilon}}^{-1})_{ql}(\sqrt{\overleftrightarrow{\epsilon}}^{-1})_{pj} + P_{iq}P_{ip}(\sqrt{\overleftrightarrow{\epsilon}}^{-1})_{qj}(\sqrt{\overleftrightarrow{\epsilon}}^{-1})_{pl}. \end{aligned} \tag{B.3}$$

For convenience, we define the tensor $\overleftrightarrow{F}$, appearing in equation (B.2), as follows:

$$F_{jl} = \sum_{i=1}^{3}\frac{1}{b_i^2+s}\left[\frac{\partial}{\partial r_j}\frac{\partial}{\partial r_l}\left(z_i^2\right)_{\vec{z}=\overleftrightarrow{P}\sqrt{\overleftrightarrow{\epsilon}}^{-1}\vec{r}}\right]. \tag{B.4}$$

It can be easily handled in tensor notation

$$\overleftrightarrow{F} = 2\sqrt{\overleftrightarrow{\epsilon}}^{-1}\overleftrightarrow{P}^T(\overleftrightarrow{b}^2+s\overleftrightarrow{I})^{-1}\overleftrightarrow{P}\sqrt{\overleftrightarrow{\epsilon}}^{-1} = 2\sqrt{\overleftrightarrow{\epsilon}}^{-1}(\overleftrightarrow{P}^T\overleftrightarrow{b}^2\overleftrightarrow{P}+s\overleftrightarrow{I})^{-1}\sqrt{\overleftrightarrow{\epsilon}}^{-1} \tag{B.5}$$

In the main text we have assumed that $\sqrt{\overleftrightarrow{\epsilon}}\overleftrightarrow{a}^{-2}\sqrt{\overleftrightarrow{\epsilon}} = \overleftrightarrow{P}^T\overleftrightarrow{b}^{-2}\overleftrightarrow{P}$ and therefore we also have $\sqrt{\overleftrightarrow{\epsilon}}^{-1}\overleftrightarrow{a}^2\sqrt{\overleftrightarrow{\epsilon}}^{-1} = \overleftrightarrow{P}^T\overleftrightarrow{b}^2\overleftrightarrow{P}$. This latter relation, placed in equation (B.5), leads to the following simple expression:

$$\overleftrightarrow{F} = 2\sqrt{\overleftrightarrow{\epsilon}}^{-1}(\sqrt{\overleftrightarrow{\epsilon}}^{-1}\overleftrightarrow{a}^2\sqrt{\overleftrightarrow{\epsilon}}^{-1}+s\overleftrightarrow{I})^{-1}\sqrt{\overleftrightarrow{\epsilon}}^{-1} = 2(\overleftrightarrow{a}^2+s\overleftrightarrow{\epsilon})^{-1}. \tag{B.6}$$

It is interesting to observe that, proceeding with the calculations, the auxiliary quantities $\overleftrightarrow{P}$ and $\overleftrightarrow{b}$, defined in equation (22), gradually disappear, obtaining the final results only in terms of $\overleftrightarrow{\epsilon}$ and $\overleftrightarrow{a}$, which are the quantities with a direct physical and geometrical meaning. Moreover, since the tensor $\overleftrightarrow{F}$ does not depend on the position $\vec{r}$, we have obtained an important achievement: the electric field generated inside the ellipsoidal inclusion is uniform. Returning to equation (B.2) for this electric field and recalling the definition given in (B.4), we obtain

$$E_j = \frac{\epsilon_{lk} E_k^*}{4} b_1 b_2 b_3 \int_0^{+\infty} F_{jl} \frac{\mathrm{d}s}{R(s)} \tag{B.7}$$

or, in tensor notation,

$$\vec{E} = \frac{1}{4} \int_0^{+\infty} \overset{\leftrightarrow}{F} \overset{\leftrightarrow}{\epsilon} \frac{\det(\overset{\leftrightarrow}{b})}{R(s)} \mathrm{d}s \vec{E}^* = \frac{1}{4} \int_0^{+\infty} 2(\overset{\leftrightarrow}{a}^2 + s\overset{\leftrightarrow}{\epsilon})^{-1} \overset{\leftrightarrow}{\epsilon} \frac{\det(\overset{\leftrightarrow}{b})}{R(s)} \mathrm{d}s \vec{E}^*. \tag{B.8}$$

From the above relation $\sqrt{\overset{\leftrightarrow}{\epsilon}}^{-1} \overset{\leftrightarrow}{a}^2 \sqrt{\overset{\leftrightarrow}{\epsilon}}^{-1} = \overset{\leftrightarrow}{P}^T \overset{\leftrightarrow}{b}^2 \overset{\leftrightarrow}{P}$ we easily obtain $\overset{\leftrightarrow}{P} \sqrt{\overset{\leftrightarrow}{\epsilon}}^{-1} \overset{\leftrightarrow}{a}^2 \sqrt{\overset{\leftrightarrow}{\epsilon}}^{-1} \overset{\leftrightarrow}{P}^T = \overset{\leftrightarrow}{b}^2$ and consequently the term $\det(\overset{\leftrightarrow}{b})/R(s)$ can be further expanded as

$$\frac{\det(\overset{\leftrightarrow}{b})}{R(s)} = \frac{\sqrt{\det(\overset{\leftrightarrow}{b}^2)}}{\sqrt{\det(\overset{\leftrightarrow}{b}^2 + s\overset{\leftrightarrow}{I})}} = \frac{\sqrt{\det(\overset{\leftrightarrow}{P} \sqrt{\overset{\leftrightarrow}{\epsilon}}^{-1} \overset{\leftrightarrow}{a}^2 \sqrt{\overset{\leftrightarrow}{\epsilon}}^{-1} \overset{\leftrightarrow}{P}^T)}}{\sqrt{\det(\overset{\leftrightarrow}{P} \sqrt{\overset{\leftrightarrow}{\epsilon}}^{-1} \overset{\leftrightarrow}{a}^2 \sqrt{\overset{\leftrightarrow}{\epsilon}}^{-1} \overset{\leftrightarrow}{P}^T + s\overset{\leftrightarrow}{I})}} = \frac{\det(\overset{\leftrightarrow}{a})}{\sqrt{\det(\overset{\leftrightarrow}{a}^2 + s\overset{\leftrightarrow}{\epsilon})}} \tag{B.9}$$

where we have repeatedly utilized the Binet theorem [38] for the product of two determinants (which can be stated in the form $\det(\overset{\leftrightarrow}{a})\det(\overset{\leftrightarrow}{b}) = \det(\overset{\leftrightarrow}{a}\overset{\leftrightarrow}{b})$ for two arbitrary tensors $\overset{\leftrightarrow}{a}$ and $\overset{\leftrightarrow}{b}$). The final result, derived from equations (B.8) and (B.9), is given in equation (29). We have found the uniform electric field $\vec{E}$ generated inside a uniform ellipsoidal inclusion ($\overset{\leftrightarrow}{a}$) in an anisotropic environment ($\overset{\leftrightarrow}{\epsilon}$) in terms of the eigenfield $\vec{E}^*$.

Appendix C. Calculation of the external field

The resulting electric field for $\vec{r} \in \Re^3 \backslash V$ is given by (see equation (24))

$$E_j(\vec{r}) = -\frac{\epsilon_{lk} E_k^*}{4\pi} \frac{\partial}{\partial r_j} \frac{\partial}{\partial r_l} [\Phi_{V'}(\vec{z})]_{\vec{z} = \overset{\leftrightarrow}{P} \sqrt{\overset{\leftrightarrow}{\epsilon}}^{-1} \vec{r}}. \tag{C.1}$$

The substitution $\vec{z} = \overset{\leftrightarrow}{P} \sqrt{\overset{\leftrightarrow}{\epsilon}}^{-1} \vec{r}$ immediately leads to the formula

$$\frac{\partial}{\partial r_j} \frac{\partial}{\partial r_l} [\Phi_{V'}(\vec{z})]_{\vec{z} = \overset{\leftrightarrow}{P} \sqrt{\overset{\leftrightarrow}{\epsilon}}^{-1} \vec{r}} = (\sqrt{\overset{\leftrightarrow}{\epsilon}}^{-1})_{jq} P_{qs}^T \frac{\partial^2 \Phi_{V'}(\vec{z})}{\partial z_s \partial z_i} P_{in} (\sqrt{\overset{\leftrightarrow}{\epsilon}}^{-1})_{nl}. \tag{C.2}$$

Now, for differentiating the harmonic potential $\Phi_{V'}(\vec{z})$ with respect to z_s and z_i we may use the generalized Leibnitz rule [39] because the argument $\vec{z}$ appears both in the lower limit and in the integrand of equation (40). The result is

$$\frac{\partial^2 \Phi_{V'}(\vec{z})}{\partial z_s \partial z_i} = \pi b_1 b_2 b_3 \left[\frac{\partial f(\vec{z}, \eta)}{\partial z_i} \frac{1}{R(\eta)} \frac{\partial \eta(\vec{z})}{\partial z_s} - \int_{\eta(\vec{z})}^{+\infty} \frac{\partial^2 f(\vec{z}, s)}{\partial z_s \partial z_i} \frac{\mathrm{d}s}{R(s)} \right]. \tag{C.3}$$

The derivative $\partial \eta(\vec{z})/\partial z_s$ can be calculated by recalling the definition of the function $\eta(\vec{z})$ (which satisfies the relation $f(\vec{z}, \eta(\vec{z})) = 1$) and by using the Dini theorem for differentiating the implicit functions [38]. So, from $f(\vec{z}, \eta(\vec{z})) = 1$ and from the definition of the function f

given in equation (27), we obtain

$$\frac{\partial f}{\partial z_s}+\frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial z_s}=0 \tag{C.4}$$

$$\frac{\partial \eta}{\partial z_s}=-\frac{\frac{\partial f}{\partial z_s}}{\frac{\partial f}{\partial \eta}}=2\frac{z_s}{b_s^2+\eta}\left(\sum_{k=1}^{3}\frac{z_k^2}{\left(b_k^2+\eta\right)^2}\right)^{-1}. \tag{C.5}$$

These derivatives are also useful to simplify equation (C.3)

$$\frac{\partial f}{\partial z_i}=2\frac{z_i}{b_i^2+s}\quad\Rightarrow\quad\frac{\partial^2 f}{\partial z_s\partial z_i}=\frac{2\delta_{si}}{b_i^2+s}. \tag{C.6}$$

Summing up, we may use equations (C.4) and (C.6) in (C.3), we insert (C.3) in (C.2) and we obtain an expanded version of equation (C.1) for the external electric field:

$$E_j(\vec{r})=\frac{b_1b_2b_3}{2}\sqrt{\overleftrightarrow{\epsilon}}^{-1}_{jq}P^T_{qs}\left[\frac{-2z_sz_i}{R(\eta)\left(b_s^2+\eta\right)\left(b_i^2+\eta\right)}\frac{1}{\sum_{k=1}^{3}\frac{z_k^2}{(b_k^2+\eta)^2}}\right.$$
$$\left.+\int_{\eta(\vec{z})}^{+\infty}\frac{\delta_{si}\,\mathrm{d}s}{R(s)\left(b_i^2+s\right)}\right]P_{in}\sqrt{\overleftrightarrow{\epsilon}}^{-1}_{nl}\epsilon_{lk}E_k^* \tag{C.7}$$

where the bracket must be calculated for $\vec{z}=\overleftrightarrow{P}\sqrt{\overleftrightarrow{\epsilon}}^{-1}\vec{r}$. As before, we wish to eliminate the auxiliary quantities $\overleftrightarrow{P}$ and $\overleftrightarrow{b}$, defined in equation (22) obtaining the final result only in terms of $\overleftrightarrow{\epsilon}$ and $\overleftrightarrow{a}$, which are the quantities with a direct meaning in our problem. We start from the definition of the function quantity $\eta(\vec{z})$, which satisfies the relation $f(\vec{z},\eta(\vec{z}))=1$. Returning to the spatial variable $\vec{r}$, we have to obtain an implicit equation for $\eta(\overleftrightarrow{P}\sqrt{\overleftrightarrow{\epsilon}}^{-1}\vec{r})$; we may proceed in this way: the relation $f(\vec{z},\eta(\vec{z}))=1$ implies that $\vec{z}^T(\overleftrightarrow{b}^2+\eta\overleftrightarrow{I})^{-1}\vec{z}=1$ and therefore we obtain the sequence of expressions

$$(\overleftrightarrow{P}\sqrt{\overleftrightarrow{\epsilon}}^{-1}\vec{r})^T(\overleftrightarrow{b}^2+\eta\overleftrightarrow{I})^{-1}\overleftrightarrow{P}\sqrt{\overleftrightarrow{\epsilon}}^{-1}\vec{r}=1\quad\Rightarrow\quad\vec{r}^T\sqrt{\overleftrightarrow{\epsilon}}^{-1}(\overleftrightarrow{P}^T\overleftrightarrow{b}^2\overleftrightarrow{P}+\eta\overleftrightarrow{I})^{-1}\sqrt{\overleftrightarrow{\epsilon}}^{-1}\vec{r}=1$$
$$\Rightarrow\quad\vec{r}^T\sqrt{\overleftrightarrow{\epsilon}}^{-1}(\sqrt{\overleftrightarrow{\epsilon}}^{-1}\overleftrightarrow{a}^2\sqrt{\overleftrightarrow{\epsilon}}^{-1}+\eta\overleftrightarrow{I})^{-1}\sqrt{\overleftrightarrow{\epsilon}}^{-1}\vec{r}=1\quad\Rightarrow\quad\vec{r}^T(\overleftrightarrow{a}^2+\eta\overleftrightarrow{\epsilon})^{-1}\vec{r}=1. \tag{C.8}$$

Hence, the function η directly depends on the main tensors $\overleftrightarrow{\epsilon}$ and $\overleftrightarrow{a}$, as expected. Now, we can begin the conversion of equation (C.7). The following term, which appears in equation (C.7), can be converted with these calculations:

$$\begin{aligned}\sum_{k=1}^{3}\frac{z_k^2}{\left(b_k^2+\eta\right)^2}&=\overbrace{(\overleftrightarrow{P}\sqrt{\overleftrightarrow{\epsilon}}^{-1}\vec{r})^T}^{\vec{z}^T}(\overleftrightarrow{b}^2+\eta\overleftrightarrow{I})^{-2}\overbrace{\overleftrightarrow{P}\sqrt{\overleftrightarrow{\epsilon}}^{-1}\vec{r}}^{\vec{z}}\\&=\vec{r}^T\sqrt{\overleftrightarrow{\epsilon}}^{-1}\overleftrightarrow{P}^T(\overleftrightarrow{b}^2+\eta\overleftrightarrow{I})^{-1}\overleftrightarrow{P}\overleftrightarrow{P}^T(\overleftrightarrow{b}^2+\eta\overleftrightarrow{I})^{-1}\overleftrightarrow{P}\sqrt{\overleftrightarrow{\epsilon}}^{-1}\vec{r}\\&=\vec{r}^T\sqrt{\overleftrightarrow{\epsilon}}^{-1}(\sqrt{\overleftrightarrow{\epsilon}}^{-1}\overleftrightarrow{a}^2\sqrt{\overleftrightarrow{\epsilon}}^{-1}+\eta\overleftrightarrow{I})^{-1}(\sqrt{\overleftrightarrow{\epsilon}}^{-1}\overleftrightarrow{a}^2\sqrt{\overleftrightarrow{\epsilon}}^{-1}+\eta\overleftrightarrow{I})^{-1}\sqrt{\overleftrightarrow{\epsilon}}^{-1}\vec{r}\\&=\vec{r}^T(\sqrt{\overleftrightarrow{\epsilon}}^{-1}\overleftrightarrow{a}^2+\eta\sqrt{\overleftrightarrow{\epsilon}})^{-1}(\overleftrightarrow{a}^2\sqrt{\overleftrightarrow{\epsilon}}^{-1}+\eta\sqrt{\overleftrightarrow{\epsilon}})^{-1}\vec{r}\\&=\vec{r}^T[(\overleftrightarrow{a}^2\sqrt{\overleftrightarrow{\epsilon}}^{-1}+\eta\sqrt{\overleftrightarrow{\epsilon}})(\sqrt{\overleftrightarrow{\epsilon}}^{-1}\overleftrightarrow{a}^2+\eta\sqrt{\overleftrightarrow{\epsilon}})]^{-1}\vec{r}\\&=\vec{r}^T(\overleftrightarrow{a}^2+\eta\overleftrightarrow{\epsilon})^{-1}\overleftrightarrow{\epsilon}(\overleftrightarrow{a}^2+\eta\overleftrightarrow{\epsilon})^{-1}\vec{r}.\end{aligned} \tag{C.9}$$

So, equation (C.7) can be converted in tensor notation as follows:

$$\vec{E}(\vec{r}) = \frac{1}{2}\det(\overleftrightarrow{b})\sqrt{\overleftrightarrow{\epsilon}}^{-1}\overleftrightarrow{P}^{T}\left[\frac{-2(\overleftrightarrow{b}^{2}+\eta\overleftrightarrow{I})^{-1}\vec{z}\vec{z}^{T}(\overleftrightarrow{b}^{2}+\eta\overleftrightarrow{I})^{-1}}{\vec{r}^{T}(\overleftrightarrow{a}^{2}+\eta\overleftrightarrow{\epsilon})^{-1}\overleftrightarrow{\epsilon}(\overleftrightarrow{a}^{2}+\eta\overleftrightarrow{\epsilon})^{-1}\vec{r}}\frac{1}{R(\eta)}\right.$$

$$\left.+\int_{\eta}^{+\infty}\frac{(\overleftrightarrow{b}^{2}+s\overleftrightarrow{I})^{-1}\,\mathrm{d}s}{R(s)}\right]\overleftrightarrow{P}\sqrt{\overleftrightarrow{\epsilon}}^{-1}\overleftrightarrow{\epsilon}\vec{E}^{*}. \tag{C.10}$$

We remember that the product $\vec{z}\vec{z}^{T}$ in the previous expression represents the external product of vectors giving, as result, a tensor $\vec{z}\vec{z}^{T}$ whose components are given by $(\vec{z}\vec{z}^{T})_{ij} = z_i z_j$. With a series of considerations similar to the previous ones we may state that

$$\sqrt{\overleftrightarrow{\epsilon}}^{-1}\overleftrightarrow{P}^{T}(\overleftrightarrow{b}^{2}+\eta\overleftrightarrow{I})^{-1}\vec{z}\vec{z}^{T}(\overleftrightarrow{b}^{2}+\eta\overleftrightarrow{I})^{-1}\overleftrightarrow{P}\sqrt{\overleftrightarrow{\epsilon}}^{-1} = (\overleftrightarrow{a}^{2}+\eta\overleftrightarrow{\epsilon})^{-1}\vec{r}\vec{r}^{T}(\overleftrightarrow{a}^{2}+\eta\overleftrightarrow{\epsilon})^{-1} \tag{C.11}$$

$$\sqrt{\overleftrightarrow{\epsilon}}^{-1}\overleftrightarrow{P}^{T}(\overleftrightarrow{b}^{2}+s\overleftrightarrow{I})^{-1}\overleftrightarrow{P}\sqrt{\overleftrightarrow{\epsilon}}^{-1} = (\overleftrightarrow{a}^{2}+\eta\overleftrightarrow{\epsilon})^{-1} \tag{C.12}$$

and, therefore, the final equation for the external electric field is given in equation (31) (we have also used equation (B.9)).

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